Smooth Mori dream spaces of small Picard number

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Introduction

This thesis contributes to the study of Mori dream spaces and their geometric aspects. Mori dream spaces, introduced by Hu and Keel [**40**], are characterized via their optimal behavior with respect to the minimal model program. Well-known example classes include projective toric varieties, smooth Fano varieties [**12**], Calabi-Yau varieties of dimension at most three having a polyhedral effective cone [**52**] and spherical varieties [**17**]. In terms of Cox rings, Mori dream spaces are characterized as the irreducible normal projective varieties X such that the divisor class group $\mathrm{Cl}(X)$ and the Cox ring

$$\mathcal{R}(X) \ := \bigoplus_{[D]\in\mathrm{Cl}(X)} \Gamma(X, \mathcal{O}_X(D))$$

are finitely generated. Similar to toric varieties, Mori dream spaces show close connections to combinatorics. They are completely described by their Cox ring and certain data from convex geometry, namely a collection of rational convex polyhedral cones in the vector space associated with the divisor class group [**11, 35, 3**]. This approach makes Mori dream spaces particularly accessible in the case of small Picard number and a Cox ring with simply structured defining relations. The latter basically means to move a controlled step beyond toric geometry. Our main results comprise classifications in the smooth case for Picard numbers up to three, including in particular new lists of smooth Fano varieties. Moreover, we provide further evidence on Mukai's conjecture and Fujita's base point free conjecture.

It is well-known that in the toric case, the only smooth projective varieties of Picard number one are the projective spaces. In Picard number two, Kleinschmidt [**47**] showed that all smooth complete toric varieties arise as projectivized split vector bundles, and Batyrev [**7**] studied the case of Picard number three via primitive collections. In Chapter two, which presents joint work with J. Hausen and M. Nicolussi [**28**], we discuss irreducible smooth projective non-toric rational varieties with a torus action of complexity one [**39, 36, 3**], i.e. the general torus orbit is of dimension one less than the variety itself. In Picard number one, the classification is due to a result of Liendo and Süß [**49**, Thm. 6.5]: there are up to isomorphism only two varieties, namely the smooth projective quadrics in dimensions three and four. In Picard number two, we obtain the following result, where we describe a variety through its Cox ring and an ample class. Note that this data determines a Mori dream space up to isomorphism; see Chapter one for details and background. As in the whole thesis, by a variety we mean a variety over an algebraically closed field $\mathbb{K}$ of characteristic zero.

Theorem 2.1.1. *Every smooth rational irreducible projective non-toric variety of Picard number two that admits a torus action of complexity one is isomorphic to precisely one of the following varieties X, specified by their Cox ring $\mathcal{R}(X)$ and an ample class $u \in \mathrm{Cl}(X)$, where we always have $\mathrm{Cl}(X) = \mathbb{Z}^2$ and the grading is fixed by the matrix $[w_1, \dots, w_r]$ of generator degrees $\deg(T_i), \deg(S_j) \in \mathrm{Cl}(X)$.*

No.	$\mathcal{R}(X)$	$[w_1,\ldots,w_r]$	u	$\dim(X)$
1	$\frac{\mathbb{K}[T_1,\ldots,T_7]}{\langle T_1T_2T_3^2+T_4T_5+T_6T_7\rangle}$	$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & a & 2-a & b & 2-b \end{bmatrix}$ $1 \leq a \leq b$	$\begin{bmatrix} 1 \\ 1+b \end{bmatrix}$	4
2	$\frac{\mathbb{K}[T_1,\ldots,T_7]}{\langle T_1T_2T_3+T_4T_5+T_6T_7\rangle}$	$\begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$	4
3	$\frac{\mathbb{K}[T_1,\ldots,T_6]}{\langle T_1T_2T_3^2+T_4T_5+T_6^2\rangle}$	$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 2-a & a & 1 \end{bmatrix}$ $a \geq 1$	$\begin{bmatrix} 1 \\ 1+a \end{bmatrix}$	3
4	$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2^{l_2}+T_3T_4^{l_4}+T_5T_6^{l_6}\rangle}$ $m\geq 0$	$\left[\begin{array}{cccccc\|ccc} 0 & 1 & a & 1 & b & 1 & c_1 & \ldots & c_m \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & \ldots & 1 \end{array}\right]$ $0 \leq a \leq b,\ c_1 \leq \ldots \leq c_m,$ $l_2 = a + l_4 = b + l_6$	$\begin{bmatrix} d+1 \\ 1 \end{bmatrix}$ $d := \max(b, c_m)$	$m+3$
5	$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2+T_3^2T_4+T_5^2T_6\rangle}$ $m\geq 0$	$\left[\begin{array}{cccccc\|ccc} 0 & 2a+1 & a & 1 & a & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & \ldots & 0 \end{array}\right]$ $a \geq 0$	$\begin{bmatrix} 2a+2 \\ 1 \end{bmatrix}$	$m+3$
6	$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2T_6\rangle}$ $m\geq 1$	$\left[\begin{array}{cccccc\|ccc} 0 & 2c+1 & a & b & c & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & \ldots & 0 \end{array}\right]$ $a, b, c \geq 0,\quad a < b,$ $a + b = 2c + 1$	$\begin{bmatrix} 2c+2 \\ 1 \end{bmatrix}$	$m+3$
7	$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$ $m\geq 1$	$\left[\begin{array}{cccccc\|ccc} 0 & 0 & 0 & 0 & -1 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & \ldots & 0 \end{array}\right]$	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$	$m+3$
8	$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$ $m\geq 2$	$\left[\begin{array}{cccccc\|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & a_2 & \ldots & a_m \end{array}\right]$ $0 \leq a_2 \leq \ldots \leq a_m,\ a_m > 0$	$\begin{bmatrix} 1 \\ a_m+1 \end{bmatrix}$	$m+3$
9	$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$ $m\geq 2$	$\left[\begin{array}{cccc\|ccc} 0 & a_2 & \ldots & a_6 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 & 0 & \ldots & 0 \end{array}\right]$ $0 \leq a_3 \leq a_5 \leq a_6 \leq a_4 \leq a_2,$ $a_2 = a_3 + a_4 = a_5 + a_6$	$\begin{bmatrix} a_2+1 \\ 1 \end{bmatrix}$	$m+3$
10	$\frac{\mathbb{K}[T_1,\ldots,T_5,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$ $m\geq 1$	$\left[\begin{array}{ccccc\|ccc} 1 & 1 & 1 & 1 & 1 & 0 & \ldots & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 & \ldots & 1 \end{array}\right]$	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$	$m+2$
11	$\frac{\mathbb{K}[T_1,\ldots,T_5,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$ $m\geq 2$	$\left[\begin{array}{ccccc\|cccc} 1 & 1 & 1 & 1 & 1 & 0 & a_2 & \ldots & a_m \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & \ldots & 1 \end{array}\right]$ $0 \leq a_2 \leq \ldots \leq a_m,\ a_m > 0$	$\begin{bmatrix} a_m+1 \\ 1 \end{bmatrix}$	$m+2$
12	$\frac{\mathbb{K}[T_1,\ldots,T_5,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$ $m\geq 2$	$\left[\begin{array}{ccccc\|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 2c & a & b & c & 1 & 1 & \ldots & 1 \end{array}\right]$ $0 \leq a \leq c \leq b,\ a + b = 2c$	$\begin{bmatrix} 1 \\ 2c+1 \end{bmatrix}$	$m+2$
13	$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\left\langle \begin{array}{c} T_1T_2+T_3T_4+T_5T_6, \\ \lambda T_3T_4+T_5T_6+T_7T_8 \end{array} \right\rangle}$ $\lambda\in\mathbb{K}^*\setminus\{1\}$	$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	4

Moreover, each of the listed data sets defines a smooth rational non-toric projective variety of Picard number two coming with a torus action of complexity one.

Toric Fano varieties are a class of varieties thoroughly investigated since the 1970s: by now, there are classification results up to dimension nine [**6, 8, 63, 48, 56, 57, 67**] in terms of their combinatorial description via lattice polytopes. Note that in case of varieties of complexity one, our Cox ring-based approach allows us to compute the anticanonical divisor class via a formula [**3**, Prop. 3.3.3.2] using the degrees of the generators and of the relations of $\mathcal{R}(X)$. In this way, we determine in every dimension the finitely many non-toric smooth rational Fano varieties of Picard number two admitting a torus action of complexity one; they are described geometrically by means of elementary contractions in Section 2.3.

Theorem 2.1.2. *Every smooth rational non-toric Fano variety of Picard number two that admits a torus action of complexity one is isomorphic to precisely one of the following varieties X, specified by their Cox ring $\mathcal{R}(X)$, where the grading by $\mathrm{Cl}(X) = \mathbb{Z}^2$ is given by the matrix $[w_1, \ldots, w_r]$ of generator degrees $\deg(T_i), \deg(S_j) \in \mathrm{Cl}(X)$ and we list the (ample) anticanonical class $-\mathcal{K}_X$.*

No.	$\mathcal{R}(X)$	$[w_1,\ldots,w_r]$	$-\mathcal{K}_X$	$\dim(X)$
1	$\frac{\mathbb{K}[T_1,\ldots,T_7]}{\langle T_1T_2T_3^2+T_4T_5+T_6T_7\rangle}$	$\begin{bmatrix}0&0&1&1&1&1&1\\1&1&0&1&1&1&1\end{bmatrix}$	$\begin{bmatrix}3\\4\end{bmatrix}$	4
2	$\frac{\mathbb{K}[T_1,\ldots,T_7]}{\langle T_1T_2T_3+T_4T_5+T_6T_7\rangle}$	$\begin{bmatrix}0&0&1&1&0&1&0\\1&1&0&1&1&1&1\end{bmatrix}$	$\begin{bmatrix}2\\4\end{bmatrix}$	4
3	$\frac{\mathbb{K}[T_1,\ldots,T_6]}{\langle T_1T_2T_3^2+T_4T_5+T_6^2\rangle}$	$\begin{bmatrix}0&0&1&1&1&1\\1&1&0&1&1&1\end{bmatrix}$	$\begin{bmatrix}2\\3\end{bmatrix}$	3
4.A	$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$ $m\geq 0$	$\left[\begin{array}{cccccc\|cccc}0&1&0&1&0&1&c&0&\ldots&0\\1&0&1&0&1&0&1&1&\ldots&1\end{array}\right]$ $c\in\{-1,0\}$, $c:=0$ *if* $m=0$	$\begin{bmatrix}2+c\\2+m\end{bmatrix}$	$m+3$
4.B	$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2^2+T_3T_4+T_5T_6\rangle}$ $m\geq 0$	$\left[\begin{array}{cccccc\|ccc}0&1&1&1&1&1&1&\ldots&1\\1&0&1&0&1&0&1&\ldots&1\end{array}\right]$	$\begin{bmatrix}3+m\\2+m\end{bmatrix}$	$m+3$
4.C	$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2^2+T_3T_4^2+T_5T_6^2\rangle}$ $m\geq 0$	$\left[\begin{array}{cccccc\|ccc}0&1&0&1&0&1&0&\ldots&0\\1&0&1&0&1&0&1&\ldots&1\end{array}\right]$	$\begin{bmatrix}1\\2+m\end{bmatrix}$	$m+3$
5	$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2+T_3^2T_4+T_5^2T_6\rangle}$ $m\geq 1$	$\left[\begin{array}{cccccc\|ccc}0&2a+1&a&1&a&1&1&\ldots&1\\1&1&1&0&1&0&0&\ldots&0\end{array}\right]$ $0\leq 2a<m$	$\begin{bmatrix}2a+m+2\\2\end{bmatrix}$	$m+3$
6	$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2T_6\rangle}$ $m\geq 2$	$\left[\begin{array}{cccccc\|ccc}0&2c+1&a&b&c&1&1&\ldots&1\\1&1&1&1&1&0&0&\ldots&0\end{array}\right]$ $a,b,c\geq 0,\quad a<b,$ $a+b=2c+1,$ $m>3c+1$	$\begin{bmatrix}3c+2+m\\3\end{bmatrix}$	$m+3$
7	$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$ $1\leq m\leq 3$	$\left[\begin{array}{cccccc\|ccc}0&0&0&0&-1&1&1&\ldots&1\\1&1&1&1&1&1&0&\ldots&0\end{array}\right]$	$\begin{bmatrix}m\\4\end{bmatrix}$	$m+3$
8	$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$ $m\geq 2$	$\left[\begin{array}{cccccc\|cccc}0&0&0&0&0&0&1&1&\ldots&1\\1&1&1&1&1&1&0&a_2&\ldots&a_m\end{array}\right]$ $0\leq a_2\leq\ldots\leq a_m,$ $a_m\in\{1,2,3\},$ $4+\sum_{k=2}^m a_k>ma_m$	$\begin{bmatrix}m\\4+\sum_{k=2}^m a_k\end{bmatrix}$	$m+3$
9	$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$ $m\geq 2$	$\left[\begin{array}{cccc\|ccc}0&a_2&\ldots&a_6&1&\ldots&1\\1&1&\ldots&1&0&\ldots&0\end{array}\right]$ $0\leq a_3\leq a_5\leq a_6\leq a_4\leq a_2,$ $a_2=a_3+a_4=a_5+a_6,$ $2a_2<m$	$\begin{bmatrix}2a_2+m\\4\end{bmatrix}$	$m+3$
10	$\frac{\mathbb{K}[T_1,\ldots,T_5,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$ $1\leq m\leq 2$	$\left[\begin{array}{ccccc\|ccc}1&1&1&1&1&0&\ldots&0\\-1&1&0&0&0&1&\ldots&1\end{array}\right]$	$\begin{bmatrix}3\\m\end{bmatrix}$	$m+2$
11	$\frac{\mathbb{K}[T_1,\ldots,T_5,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$ $m\geq 2$	$\left[\begin{array}{ccccc\|cccc}1&1&1&1&1&0&a_2&\ldots&a_m\\0&0&0&0&0&1&1&\ldots&1\end{array}\right]$ $0\leq a_2\leq\ldots\leq a_m,$ $a_m\in\{1,2\},$ $3+\sum_{k=2}^m a_k>ma_m$	$\begin{bmatrix}3+\sum_{k=2}^m a_k\\m\end{bmatrix}$	$m+2$
12	$\frac{\mathbb{K}[T_1,\ldots,T_5,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$ $m\geq 2$	$\left[\begin{array}{ccccc\|cccc}1&1&1&1&1&0&0&\ldots&0\\0&2c&a&b&c&1&1&\ldots&1\end{array}\right]$ $0\leq a\leq c\leq b,\ a+b=2c,$ $3c<m$	$\begin{bmatrix}3\\3c+m\end{bmatrix}$	$m+2$
13	$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\left\langle\begin{array}{c}T_1T_2+T_3T_4+T_5T_6,\\ \lambda T_3T_4+T_5T_6+T_7T_8\end{array}\right\rangle}$ $\lambda\in\mathbb{K}^*\setminus\{1\}$	$\begin{bmatrix}1&0&1&0&1&0&1&0\\0&1&0&1&0&1&0&1\end{bmatrix}$	$\begin{bmatrix}2\\2\end{bmatrix}$	4

Moreover, each of the listed data sets defines a smooth rational non-toric Fano variety of Picard number two coming with a torus action of complexity one.

It turns out that the varieties of Theorem 2.1.2 are obtained from varieties Y with dimension at most seven via duplication of some of the free weights of Cox rings $\mathcal{R}(Y)$, i.e. given a variable that does not show up in the defining trinomials, one adds a further free variable of the same degree. The geometric interpretation of this procedure is the following: one takes a certain $\mathbb{P}_1$-bundle over the original variety Y, applies a natural series of flips and then contracts a prime divisor, see Section 2.2 for details.

Corollary 2.1.3. *Every smooth rational non-toric Fano variety with a torus action of complexity one and Picard number two arises via iterated duplication of a free weight from a smooth rational projective (not necessarily Fano) variety with a torus action of complexity one, Picard number two and dimension at most seven.*

Jahnke, Peternell and Radloff [**42, 43**] obtained a classification of smooth threefolds of Picard number two that are almost Fano, i.e. whose anticanonical divisor is big and nef. Note that in general, the problem of describing smooth almost Fano varieties is widely open. In the setting of a torus action of complexity one, we may – as in the Fano case – figure out the non-toric rational smooth almost Fano varieties in arbitrary dimension. Together with Theorem 2.1.2, the following result classifying *truly almost Fano varieties*, i.e. varieties that are almost Fano but not Fano, settles the description.

Theorem 2.1.4. *Every smooth rational non-toric truly almost Fano variety of Picard number two that admits a torus action of complexity one is isomorphic to precisely one of the following varieties X, specified by their Cox ring $\mathcal{R}(X)$ and an ample class $u \in \mathrm{Cl}(X)$, where we always have $\mathrm{Cl}(X) = \mathbb{Z}^2$ and the grading is fixed by the matrix $[w_1, \ldots, w_r]$ of generator degrees $\deg(T_i), \deg(S_j) \in \mathrm{Cl}(X)$.*

No.	$\mathcal{R}(X)$	$[w_1, \ldots, w_r]$	u	$\dim(X)$
4.A	$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$ $m\geq 1$	$\left[\begin{array}{cccccc\|ccc} 0&1&0&1&0&1&c_1&\ldots&c_m \\ 1&0&1&0&1&0&1&\ldots&1 \end{array}\right]$ $c_1 \leq \ldots \leq c_m$ $d := \max(0, c_m)$ $(2+m)d = 2 + c_1 + \cdots + c_m$	$\begin{bmatrix} 1 \\ 1+d \end{bmatrix}$	$m+3$
4.B	$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2^2+T_3T_4+T_5T_6\rangle}$ $m\geq 1$	$\left[\begin{array}{cccccc\|cccc} 0&1&1&1&1&1&0&1&\ldots&1 \\ 1&0&1&0&1&0&1&1&\ldots&1 \end{array}\right]$	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$	$m+3$
4.C	$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2^2+T_3T_4^2+T_5T_6^2\rangle}$ $m\geq 1$	$\left[\begin{array}{cccccc\|cccc} 0&1&0&1&0&1&-1&0&\ldots&0 \\ 1&0&1&0&1&0&1&1&\ldots&1 \end{array}\right]$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$m+3$
4.D	$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2^2+T_3T_4^2+T_5T_6\rangle}$ $m\geq 0$	$\left[\begin{array}{cccccc\|ccc} 0&1&0&1&1&1&1&\ldots&1 \\ 1&0&1&0&1&0&1&\ldots&1 \end{array}\right]$	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$	$m+3$
4.E	$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2^3+T_3T_4+T_5T_6\rangle}$ $m\geq 0$	$\left[\begin{array}{cccccc\|ccc} 0&1&2&1&2&1&2&\ldots&2 \\ 1&0&1&0&1&0&1&\ldots&1 \end{array}\right]$	$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$	$m+3$
4.F	$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2^3+T_3T_4^2+T_5T_6^2\rangle}$ $m\geq 0$	$\left[\begin{array}{cccccc\|ccc} 0&1&1&1&1&1&1&\ldots&1 \\ 1&0&1&0&1&0&1&\ldots&1 \end{array}\right]$	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$	$m+3$
5	$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2+T_3^2T_4+T_5^2T_6\rangle}$ $m\geq 0$	$\left[\begin{array}{cccccc\|ccc} 0&2a+1&a&1&a&1&1&\ldots&1 \\ 1&1&1&0&1&0&0&\ldots&0 \end{array}\right]$ $m = 2a$	$\begin{bmatrix} m+2 \\ 1 \end{bmatrix}$	$m+3$
6	$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2T_6\rangle}$ $m\geq 1$	$\left[\begin{array}{cccccc\|ccc} 0&2c+1&a&b&c&1&1&\ldots&1 \\ 1&1&1&1&1&0&0&\ldots&0 \end{array}\right]$ $a, b, c \geq 0, \quad a < b,$ $a + b = 2c + 1,$ $m = 3c + 1$	$\begin{bmatrix} 2c+2 \\ 1 \end{bmatrix}$	$m+3$
7	$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$ $m=4$	$\left[\begin{array}{cccccc\|cccc} 0&0&0&0&-1&1&1&1&1&1 \\ 1&1&1&1&1&1&0&0&0&0 \end{array}\right]$	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$	7
8	$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$ $m\geq 2$	$\left[\begin{array}{cccccc\|cccc} 0&0&0&0&0&0&1&1&\ldots&1 \\ 1&1&1&1&1&1&0&a_2&\ldots&a_m \end{array}\right]$ $0 \leq a_2 \leq \ldots \leq a_m, \ a_m > 0,$ $4 + a_2 + \ldots + a_m = m a_m$	$\begin{bmatrix} 1 \\ a_m+1 \end{bmatrix}$	$m+3$
9	$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$ $m\geq 2$	$\left[\begin{array}{cccc\|ccc} 0&a_2&\ldots&a_6&1&\ldots&1 \\ 1&1&\ldots&1&0&\ldots&0 \end{array}\right]$ $0 \leq a_3 \leq a_5 \leq a_6 \leq a_4 \leq a_2,$ $a_2 = a_3 + a_4 = a_5 + a_6,$ $m = 2a_2$	$\begin{bmatrix} a_2+1 \\ 1 \end{bmatrix}$	$m+3$

10	$\frac{\mathbb{K}[T_1,\dots,T_5,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$ $m=3$	$\left[\begin{array}{ccccc\|ccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right]$	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$	5
11	$\frac{\mathbb{K}[T_1,\dots,T_5,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$ $m\geq 2$	$\left[\begin{array}{ccccc\|cccc} 1 & 1 & 1 & 1 & 1 & 0 & a_2 & \dots & a_m \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & \dots & 1 \end{array}\right]$ $0 \leq a_2 \leq \dots \leq a_m,\ a_m > 0,$ $3 + a_2 + \dots + a_m = ma_m$	$\begin{bmatrix} 1 \\ a_m + 1 \end{bmatrix}$	$m+2$
12	$\frac{\mathbb{K}[T_1,\dots,T_5,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$ $m\geq 3$	$\left[\begin{array}{ccccc\|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & \dots & 0 \\ 0 & 2c & a & b & c & 1 & 1 & \dots & 1 \end{array}\right]$ $0 \leq a \leq c \leq b,\ a + b = 2c,$ $m = 3c$	$\begin{bmatrix} 1 \\ 2c + 1 \end{bmatrix}$	$m+2$

Moreover, each of the listed data sets defines a smooth rational non-toric truly almost Fano variety of Picard number two coming with a torus action of complexity one.

In Chapter three we consider a possibility to move beyond toric geometry other than the one chosen in Chapter two: We study *intrinsic quadrics*, i.e. irreducible normal projective varieties X with finitely generated divisor class group and finitely generated Cox ring $\mathcal{R}(X)$ admitting homogeneous generators such that $\mathcal{R}(X)$ is the factor ring of a polynomial ring and an ideal generated by a single homogeneous purely quadratic polynomial. For further research on intrinsic quadrics see **[11]** and **[14]**. Similar to the toric case, in Picard number one, we show that there is just one smooth projective intrinsic quadric per dimension.

Proposition 3.2.1. *Let X be a smooth intrinsic quadric of Picard number one. Then X is isomorphic to the variety defined by the Cox ring*

$$\mathbb{K}[T_1, \dots, T_r]/\langle T_1T_2 + T_3T_4 + \dots + T_{i-1}T_i + h\rangle,$$

where $i = r - 2$, $h = T_{r-1}T_r$ or $i = r - 1$, $h = T_r^2$ holds, and where the grading is given by $\deg(T_j) = 1 \in \mathbb{Z} = \mathrm{Cl}(X)$ for all $1 \leq j \leq r$. In particular, X is Fano.

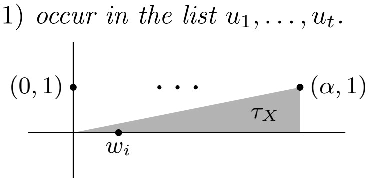
In Picard number two, Theorem 3.2.8 provides a classification of all smooth projective intrinsic quadrics, thereby generalizing a result of **[11]** that described the case of full intrinsic quadrics, i.e. the case of intrinsic quadrics whose Cox ring admits no generators that do not show up in the defining quadratic polynomial.

Theorem 3.2.8. *Every smooth intrinsic quadric of Picard number two is isomorphic to a variety X with Cox ring given by $\mathcal{R}(X) = \mathbb{K}[T_1, \dots, T_r, S_1, \dots, S_t]/\langle g\rangle$, where*

$$g \ := \ \begin{cases} T_1T_2 + \dots + T_{r-1}T_r & \text{if } r \text{ is even,} \\ T_1T_2 + \dots + T_{r-2}T_{r-1} + T_r^2 & \text{if } r \text{ is odd,} \end{cases}$$

holds for some integers $r \in \mathbb{Z}_{\geq 5}$ and $t \in \mathbb{Z}_{\geq 0}$. Furthermore, the $\mathrm{Cl}(X) = \mathbb{Z}^2$-grading of $\mathcal{R}(X)$ is obtained by choosing weights $w_i = \deg(T_i)$ and $u_j = \deg(S_j)$ according to one of the following settings, where the semiample cone τ_X of X is as indicated in the below figures.

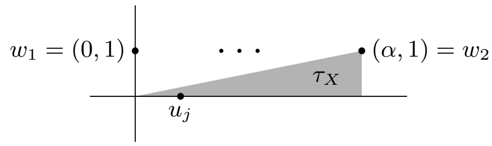
Setting 1: *Fix $\alpha \in \mathbb{Z}_{\geq 0}$. The weights u_j are taken from $(a, 1)$, where $0 \leq a \leq \alpha$ holds and we have $w_i = (1, 0)$ for all $1 \leq i \leq r$. Furthermore, we have $t \geq 2$ and the vectors $(\alpha, 1)$ and $(0, 1)$ occur in the list $u_1, \dots, u_t$.*



If X arises from Setting 1, then X is smooth and admits an elementary contraction of fiber type $\varphi\colon X \to V_{\mathbb{P}_{r-1}}(g)$ with fibers isomorphic to $\mathbb{P}_{t-1}$.

Setting 2: *Fix $\alpha \in \mathbb{Z}_{\geq 0}$. The weights w_i are taken from $(a, 1)$, where $0 \leq a \leq \alpha$ holds and we have $u_j = (1, 0)$ for all $1 \leq j \leq t$. Furthermore, we have $t \geq 2$ and the weights satisfy*

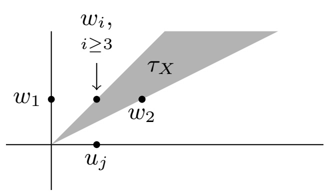
(i) $w_1 = (0, 1)$ *and* $w_2 = (\alpha, 1)$,
(ii) $w_i + w_{i+1} = (\alpha, 2)$ *for all odd* $i < r$ *and* $2w_r = (\alpha, 2)$ *if* r *is odd.*



If X arises from Setting 2, then X is smooth and admits an elementary contraction of fiber type $\varphi\colon X \to \mathbb{P}_{t-1}$ with fibers isomorphic to $V_{\mathbb{P}_{r-1}}(g)$.

Setting 3: *The weights w_i and u_j satisfy*

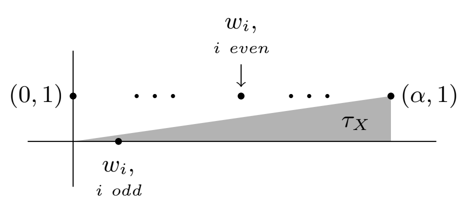
(i) $w_1 = (0, 1)$ *and* $w_2 = (2, 1)$,
(ii) $w_i = (1, 1)$ *for all* $3 \leq i \leq r$,
(iii) $u_j = (1, 0)$ *for all* $1 \leq j \leq t$ *and we have* $t \geq 1$.



If X arises from Setting 3, then X is smooth and admits an elementary birational divisorial contraction $\varphi\colon X \to \mathbb{P}_{r+t-3}$ with center isomorphic to $V_{\mathbb{P}_{r-3}}(g - T_1T_2)$.

Setting 4: *Here, $r \in \mathbb{Z}_{\geq 6}$ is even. The weights u_j are taken from $(a, 1)$, where $0 \leq a \leq \alpha$ holds with some $\alpha \in \mathbb{Z}_{\geq 0}$. We have $w_1 = (1, 0)$ and $w_2 = (w_2^1, 1)$ for some $0 \leq w_2^1 \leq \alpha$. Furthermore the weights satisfy*

(i) $w_i = w_1$ *for all odd* $1 \leq i \leq r-1$ *and* $w_i = w_2$ *for all even* $2 \leq i \leq r$,
(ii) *the vectors* $(\alpha, 1)$ *and* $(0, 1)$ *occur in the list* $w_1, \ldots, w_r, u_1, \ldots, u_t$.



If X arises from Setting 4, then X is smooth and admits an elementary contraction of fiber type $\varphi\colon X \to \mathbb{P}_{r/2-1}$ with fibers isomorphic to $\mathbb{P}_{r/2+t-2}$.

Note that the full smooth intrinsic quadrics of Picard number two described in **[11]** are precisely the examples with $\alpha = m = 0$ in Setting 4 of the above theorem. Moreover, the cases $n = 5$ and $n = 6$ in Settings 1 to 4 of Theorem 3.2.8 are the ones allowing a torus action of complexity one and thus are exactly the overlap with the description presented in Chapter two. As in our classification of varieties with a torus action of complexity one, we compute the anticanonical class and in this way derive the Fano and the truly almost Fano varieties among all smooth intrinsic quadrics of Picard number two, see Theorem 3.2.10 and Theorem 3.2.11. As an application, we prove that smooth Fano intrinsic quadrics of Picard number two fulfill Mukai's conjecture, see Proposition 3.2.14.

Having studied intrinsic quadrics of Picard number two, we go one step further and investigate smooth intrinsic quadrics of Picard number three. In Theorem 3.3.2, we provide a complete description of the smooth projective full intrinsic quadrics of Picard number three in arbitrary dimension. It turns out that there are no Fano varieties in this case. We obtain the following corollary:

Corollary 3.3.3. *Let X be a smooth full intrinsic quadric. If X is Fano, then the Picard number of X is at most two. In particular, X then is isomorphic to one of the varieties of Proposition 3.2.1 or of Setting 4 in Theorem 3.2.10 with $\alpha = t = 0$.*

In general, it turns out that the case of Picard number three is considerably larger than the case of Picard number two: Specializing to dimension at most three we obtain in Theorem 3.3.5 five series of varieties, i.e. five collections of infinitely many varieties whose Cox rings are defined by the same relation but integer parameters are allowed in the degrees of the generators, plus one sporadic variety, i.e. a single variety fitting not into the other series. In dimension four, we obtain 31 series plus six sporadic varieties, all of them listed in the table of the following theorem, where the sporadic varieties are Nos. 5, 15 and 34–37.

Theorem 3.3.6. *Every smooth intrinsic quadric of Picard number three and dimension four is isomorphic to one of the following varieties X, specified by their Cox ring $\mathcal{R}(X)$ and their semiample cone $\mathrm{SAmple}(X)$, where we always have $\mathrm{Cl}(X) = \mathbb{Z}^3$ and the grading is fixed by the matrix $Q = [w_1, \dots, w_8]$ of generator degrees $w_i = \deg(T_i) \in \mathrm{Cl}(X)$. If not indicated otherwise, the letters a, b and c denote arbitrary integers.*

No.	$\mathcal{R}(X)$	$Q = [w_1, \dots, w_8]$	$\mathrm{SAmple}(X)$ *is the the intersection of the following cones*
1	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7T_8\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|cc} 1 & a-1 & 0 & a & 0 & a & 1 & a-1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$ $a \geq 0$	$\mathrm{cone}(w_1, w_6, w_4 + w_6)$
2	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & -1 & 0 & 0 & 0 & 0 & a & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & b & c \end{array}\right]$ $b \leq 0,\ c < 0$	$\mathrm{cone}(w_1, w_3, w_5)$, $\mathrm{cone}(w_1, w_5, w_7)$, $\mathrm{cone}(w_2, w_5, w_8)$, $\mathrm{cone}(w_4, w_7, w_8)$
3	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,S_2]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & a \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right]$	$\mathrm{cone}(w_1, w_5, w_7)$, $\mathrm{cone}(w_2, w_3, w_8)$
4	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 & 1 & 0 & a & 0 \end{array}\right]$	$\mathrm{cone}(w_1, w_3, w_4)$, $\mathrm{cone}(w_2, w_7, w_8)$
5	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & -2 \\ 0 & 1 & 0 & 1 & 1 & 0 & -1 & 1 \end{array}\right]$	$\mathrm{cone}(w_1, w_5, w_7)$, $\mathrm{cone}(w_1, w_6, w_8)$
6	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & a & b \end{array}\right]$ $0 > a \geq b$	$\mathrm{cone}(w_1, w_3, w_5)$, $\mathrm{cone}(w_2, w_5, w_7)$
7	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 1 & 0 & a & 0 \end{array}\right]$ $a < 0$	$\mathrm{cone}(w_1, w_3, w_5)$
8	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & a \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & -1 & 1 \end{array}\right]$	$\mathrm{cone}(w_1, w_3, w_5)$, $\mathrm{cone}(w_1, w_7, w_8)$
9	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & a-1 & 0 & a & 0 & a & b & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right]$ $a \geq 0$	$\mathrm{cone}(w_1, w_4, w_6)$, $\mathrm{cone}(w_2, w_6, w_8)$, $\mathrm{cone}(w_4, w_7, w_8)$
10	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,S_2]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & a-1 & 0 & a & 0 & a & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right]$ $a > 0$	$\mathrm{cone}(w_1, w_4, w_6)$, $\mathrm{cone}(w_2, w_6, w_7)$

11	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & a-1 & 0 & a & 0 & a & 1 & b \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & c \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$ $a \geq 0$	$\mathrm{cone}(w_1, w_6, w_8)$, $\mathrm{cone}(w_2, w_6, w_7)$, $\mathrm{cone}(w_4, w_6, w_7)$
12	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & a & 0 \\ 0 & -1 & 0 & -1 & 0 & -1 & 1 & 1 \end{array}\right]$	$\mathrm{cone}(w_2, w_3, w_7)$, $\mathrm{cone}(w_1, w_3, w_8)$
13	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & 0 & 0 & 1 & 1 & 0 & a & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$	$\mathrm{cone}(w_1, w_3, w_7)$, $\mathrm{cone}(w_1, w_3, w_8)$
14	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & a & 0 & a & b & a-b & 1 & 1 \end{array}\right]$	$\mathrm{cone}(w_1, w_6, w_7)$, $\mathrm{cone}(w_2, w_4, w_7)$ $\mathrm{cone}(w_2, w_5, w_7)$, $\mathrm{cone}(w_3, w_5, w_7)$ $\mathrm{cone}(w_4, w_6, w_7)$, $\mathrm{cone}(w_1, w_3, w_7)$
15	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 & 1 & 1 & -1 & 0 \\ 0 & -2 & 0 & -2 & -1 & -1 & 1 & 1 \end{array}\right]$	$\mathrm{cone}(w_1, w_3, w_7)$, $\mathrm{cone}(w_1, w_3, w_8)$
16	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & 1 & 0 & 2 & 1 & 1 & 0 & 0 \\ 0 & a & 0 & a & b & a-b & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \end{array}\right]$	$\mathrm{cone}(w_1, w_4, w_8)$, $\mathrm{cone}(w_1, w_7, w_8)$, $\mathrm{cone}(w_2, w_4, w_8)$, $\mathrm{cone}(w_2, w_7, w_8)$, $\mathrm{cone}(w_5, w_4, w_8)$, $\mathrm{cone}(w_5, w_7, w_8)$, $\mathrm{cone}(w_6, w_4, w_8)$, $\mathrm{cone}(w_6, w_7, w_8)$
17	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & 1 & 0 & 2 & 1 & 1 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$	$\mathrm{cone}(w_1, w_3, w_7)$, $\mathrm{cone}(w_1, w_4, w_8)$, $\mathrm{cone}(w_1, w_7, w_8)$
18	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & 1 & 0 & 2 & 1 & 1 & 0 & -2 \\ a & 1 & 0 & 1+a & a+b & 1-b & 1 & -a \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$	$\mathrm{cone}(w_1, w_3, w_7)$, $\mathrm{cone}(w_1, w_7, w_8)$, $\mathrm{cone}(w_2, w_3, w_7)$, $\mathrm{cone}(w_2, w_7, w_8)$, $\mathrm{cone}(w_5, w_3, w_7)$, $\mathrm{cone}(w_5, w_7, w_8)$, $\mathrm{cone}(w_6, w_3, w_7)$, $\mathrm{cone}(w_6, w_7, w_8)$
19	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|ccc} 1 & -1 & 0 & 0 & 0 & 0 & a & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & b & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$	$\mathrm{cone}(w_1, w_3, w_6)$, $\mathrm{cone}(w_1, w_3, w_7)$
20	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|ccc} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & a & b \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right]$	$\mathrm{cone}(w_1, w_3, w_6)$, $\mathrm{cone}(w_1, w_3, w_7)$, $\mathrm{cone}(w_2, w_3, w_8)$, $\mathrm{cone}(w_3, w_7, w_8)$
21	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|ccc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & a & b \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right]$ $a \geq b$	$\mathrm{cone}(w_1, w_3, w_6)$, $\mathrm{cone}(w_2, w_3, w_7)$, $\mathrm{cone}(w_3, w_6, w_7)$
22	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|ccc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & a & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$ $a \neq -2$	$\mathrm{cone}(w_1, w_3, w_6)$, $\mathrm{cone}(w_2, w_3, w_7)$
23	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|ccc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & a-1 \\ 0 & 2 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & a \end{array}\right]$	$\mathrm{cone}(w_1, w_7, w_8)$, $\mathrm{cone}(w_2, w_3, w_7)$, $\mathrm{cone}(w_3, w_6, w_7)$, $\mathrm{cone}(w_3, w_7, w_8)$
24	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|ccc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 2 & 1 & 1 & 1 & 0 & a & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$	$\mathrm{cone}(w_1, w_3, w_6)$, $\mathrm{cone}(w_2, w_3, w_7)$
25	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|ccc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & a \\ 0 & 2 & 1 & 1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right]$	$\mathrm{cone}(w_1, w_3, w_6)$, $\mathrm{cone}(w_1, w_7, w_8)$
26	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|ccc} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & 0 & -1 \\ 0 & a & 0 & a & a/2 & 1 & 1 & b \end{array}\right]$ $a \in 2\mathbb{Z},\ a \leq 0$	$\mathrm{cone}(w_1, w_3, w_7)$, $\mathrm{cone}(w_2, w_6, w_8)$, $\mathrm{cone}(w_3, w_6, w_8)$
27	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|ccc} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & 0 & -2 \\ 0 & a & 0 & a & a/2 & 1 & 1 & 1-a \end{array}\right]$ $a \in 2\mathbb{Z},\ a < 0$	$\mathrm{cone}(w_1, w_3, w_6)$, $\mathrm{cone}(w_3, w_6, w_8)$
28	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|ccc} 1 & -1 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & a & 0 & a & a/2 & 1 & 1 & 1 \end{array}\right]$ $a \in 2\mathbb{Z},\ a < 0$	$\mathrm{cone}(w_2, w_3, w_8)$, $\mathrm{cone}(w_3, w_6, w_8)$
29	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|ccc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & -2 & -2 \\ 0 & a & 0 & a & a/2 & 1 & 1-a & 1-a \end{array}\right]$ $a \in 2\mathbb{Z},\ a \leq 0$	$\mathrm{cone}(w_1, w_3, w_6)$, $\mathrm{cone}(w_3, w_6, w_7)$
30	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|ccc} 1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & a & 0 & a & a/2 & 1 & 1 & 1 \end{array}\right]$ $a \in 2\mathbb{Z},\ a \leq 0$	$\mathrm{cone}(w_2, w_3, w_7)$, $\mathrm{cone}(w_3, w_6, w_7)$

31	$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|ccc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & -2 & -1 \\ 0 & a & 0 & a & a/2 & 1 & 1-a & b \end{array}\right]$ $a \in 2\mathbb{Z},\ a \le 0$	$\mathrm{cone}(w_1,w_3,w_6),\ \mathrm{cone}(w_2,w_6,w_8),$ $\mathrm{cone}(w_3,w_6,w_7),\ \mathrm{cone}(w_3,w_6,w_8)$
32	$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|ccc} 1 & -1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 2 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & a & 0 & a & a/2 & 1 & 1 & b \end{array}\right]$ $a \in 2\mathbb{Z},\ a \le 0$	$\mathrm{cone}(w_1,w_7,w_8),\ \mathrm{cone}(w_2,w_3,w_7),$ $\mathrm{cone}(w_3,w_6,w_7),\ \mathrm{cone}(w_3,w_7,w_8)$
33	$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|ccc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 2 & 1 & 1 & 1 & 0 & -1 & 1 \\ 0 & a & 0 & a & a/2 & 1 & b & 1-b \end{array}\right]$ $a \in 2\mathbb{Z},\ a \le 0$	$\mathrm{cone}(w_1,w_3,w_6),\ \mathrm{cone}(w_1,w_6,w_8),$ $\mathrm{cone}(w_2,w_7,w_8),\ \mathrm{cone}(w_3,w_6,w_7)$
34	$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|ccc} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & 0 & -1 \\ 0 & -2 & 0 & -2 & -1 & 1 & 1 & 1 \end{array}\right]$	$\mathrm{cone}(w_1,w_3,w_6),\ \mathrm{cone}(w_1,w_3,w_8)$
35	$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|ccc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & -1 & -1 \\ 0 & -2 & 0 & -2 & -1 & 1 & 1 & 1 \end{array}\right]$	$\mathrm{cone}(w_1,w_3,w_6),\ \mathrm{cone}(w_1,w_3,w_8)$
36	$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|ccc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & -1 & -1 \\ 0 & -2 & 0 & -2 & -1 & 1 & 1 & 0 \end{array}\right]$	$\mathrm{cone}(w_1,w_3,w_6),\ \mathrm{cone}(w_1,w_3,w_7)$
37	$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|ccc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 2 & 1 & 1 & 1 & 0 & -1 & 1 \\ 0 & -2 & 0 & -2 & -1 & 1 & 1 & 0 \end{array}\right]$	$\mathrm{cone}(w_1,w_3,w_6),\ \mathrm{cone}(w_1,w_3,w_7)$

Moreover, each of the listed data sets defines a smooth intrinsic quadric of Picard number three and dimension four.

We also determine the smooth Fano and the smooth almost Fano intrinsic quadrics of Picard number three and dimension at most four; see Theorem 3.3.5 for the three-dimensional case and the following theorems for the case of dimension four. It turns out that all smooth Fano intrinsic quadrics of dimension at most four and Picard number three admit a torus action of complexity one and that there is exactly one smooth almost Fano intrinsic quadric of dimension four and Picard number three that is not a complexity one T-variety, see No. 1 in Theorem 3.3.10. In order to provide a comprehensive description of our classification results for smooth projective Fano intrinsic quadrics in Picard number three and dimension four, we give in Section 3.4 a geometric interpretation in terms of elementary contractions.

Theorem 3.3.8. *Every smooth Fano intrinsic quadric of Picard number three and dimension four is isomorphic to one of the following varieties X, specified by their Cox ring $\mathcal{R}(X)$ and their semiample cone* $\mathrm{SAmple}(X)$*, where we always have* $\mathrm{Cl}(X) = \mathbb{Z}^3$ *and the grading is fixed by the matrix $Q = [w_1, \ldots, w_8]$ of generator degrees $w_i = \deg(T_i) \in \mathrm{Cl}(X)$.*

No.	$\mathcal{R}(X)$	$Q = [w_1, \ldots, w_8]$	$-\mathcal{K}_X$
2	$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & -1 \end{array}\right]$	$\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$
3	$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,S_2]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & a \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right]$ $-2 \le a \le 0$	$\begin{bmatrix} 1 \\ 3+a \\ 2 \end{bmatrix}$
4	$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 & 1 & 0 & a & 0 \end{array}\right]$ $-1 \le a \le 0$	$\begin{bmatrix} 1 \\ 2 \\ 2+a \end{bmatrix}$
7	$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 1 & 0 & -1 & 0 \end{array}\right]$	$\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$
9	$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right]$	$\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$

13, 14	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & 0 & 0 & 1 & 1 & 0 & a & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$ $-1 \leq a, b \leq 1$	$\begin{bmatrix} 2+a \\ 2+b \\ 2 \end{bmatrix}$
16	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & 1 & 0 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \end{array}\right]$	$\begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}$
17, 18	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & 1 & 0 & 2 & 1 & 1 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$ $-3 \leq a \leq 1$	$\begin{bmatrix} 4+a \\ 2 \\ 1 \end{bmatrix}$
19	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|ccc} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & a & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$ $-1 \leq a \leq 1$	$\begin{bmatrix} 1 \\ 2+a \\ 2 \end{bmatrix}$
20, 21, 30	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|ccc} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right]$ $-2 \leq a \leq -1$	$\begin{bmatrix} 1 \\ 3+a \\ 3 \end{bmatrix}$
26	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|ccc} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & a \end{array}\right]$ $-1 \leq a \leq 0$	$\begin{bmatrix} 1 \\ 2 \\ 2+a \end{bmatrix}$

Moreover, each of the listed data defines a smooth Fano intrinsic quadric of Picard number three and dimension four.

Theorem 3.3.10. *Every smooth truly almost Fano intrinsic quadric of Picard number three and dimension four is isomorphic to one of the following varieties X, specified by their Cox ring $\mathcal{R}(X)$ and their semiample cone $\mathrm{SAmple}(X)$, where we always have $\mathrm{Cl}(X) = \mathbb{Z}^3$ and the grading is fixed by the matrix $Q = [w_1, \dots, w_8]$ of generator degrees $w_i = \deg(T_i) \in \mathrm{Cl}(X)$.*

No.	$\mathcal{R}(X)$	$Q = [w_1, \dots, w_8]$	$\mathrm{SAmple}(X)$ *is the intersection of the following cones*
1	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7T_8\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|cc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$	$\mathrm{cone}(w_1, w_6, w_4 + w_6)$
2	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & -1 & 0 & 0 & 0 & 0 & a & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & b & c \end{array}\right]$ $-1 \leq a \leq 0,\ b = -1, c = -1$ *or* $-1 \leq a \leq 0,\ b = 0, c = -2$ *or* $-1 \leq a \leq 0,\ b = 1, c = 0$ *or* $a = -1, b = 0, c = -1$	$\mathrm{cone}(w_1, w_3, w_5)$, $\mathrm{cone}(w_1, w_5, w_7)$, $\mathrm{cone}(w_2, w_5, w_8)$, $\mathrm{cone}(w_4, w_7, w_8)$
3	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,S_2]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & -3 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right]$	$\mathrm{cone}(w_1, w_5, w_7)$, $\mathrm{cone}(w_2, w_3, w_8)$
4	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 & 1 & 0 & a & 0 \end{array}\right]$ $a = 1$ *or* $a = -2$	$\mathrm{cone}(w_1, w_3, w_4)$, $\mathrm{cone}(w_2, w_7, w_8)$
6	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & -1 & -1 \end{array}\right]$	$\mathrm{cone}(w_1, w_3, w_5)$, $\mathrm{cone}(w_2, w_5, w_7)$
7	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 1 & 0 & -2 & 0 \end{array}\right]$	$\mathrm{cone}(w_1, w_3, w_5)$
8	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & a \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & -1 & 1 \end{array}\right]$ $-1 \leq a \leq 0$	$\mathrm{cone}(w_1, w_3, w_5)$, $\mathrm{cone}(w_1, w_7, w_8)$
9	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & -1 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right]$	$\mathrm{cone}(w_1, w_4, w_6)$, $\mathrm{cone}(w_2, w_6, w_8)$, $\mathrm{cone}(w_4, w_7, w_8)$
10	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,S_2]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right]$	$\mathrm{cone}(w_1, w_4, w_6)$, $\mathrm{cone}(w_2, w_6, w_7)$

11	$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & b \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$ $-1 \leq b \leq 0$	$\mathrm{cone}(w_1, w_6, w_8)$, $\mathrm{cone}(w_2, w_6, w_7)$, $\mathrm{cone}(w_4, w_6, w_7)$
12	$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & a & 0 \\ 0 & -1 & 0 & -1 & 0 & -1 & 1 & 1 \end{array}\right]$ $-2 \leq a \leq -1$	$\mathrm{cone}(w_2, w_3, w_7)$, $\mathrm{cone}(w_1, w_3, w_8)$
13	$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & 0 & 0 & 1 & 1 & 0 & a & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$ $a = \pm 2,\ -2 \leq b \leq 2$ $or\ b = \pm 2,\ -1 \leq a \leq 1$	$\mathrm{cone}(w_1, w_3, w_7)$, $\mathrm{cone}(w_1, w_3, w_8)$
14	$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & b & -1-b & 1 & 1 \end{array}\right]$ $a = 1,\ 0 \leq b \leq 1$ $or\ a = 0,\ b = \pm 1$ $or\ a = -1,\ -1 \leq b \leq 0$	$\mathrm{cone}(w_1, w_6, w_7)$, $\mathrm{cone}(w_2, w_4, w_7)$ $\mathrm{cone}(w_2, w_5, w_7)$, $\mathrm{cone}(w_3, w_5, w_7)$ $\mathrm{cone}(w_4, w_6, w_7)$, $\mathrm{cone}(w_1, w_3, w_7)$
17	$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & 1 & 0 & 2 & 1 & 1 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$ $a = -4\ or\ a = 2$	$\mathrm{cone}(w_1, w_3, w_7)$, $\mathrm{cone}(w_1, w_4, w_8)$, $\mathrm{cone}(w_1, w_7, w_8)$
19	$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|ccc} 1 & -1 & 0 & 0 & 0 & 0 & a & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & b & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$ $a = \pm 1,\ -2 \leq b \leq 2$ $or\ a = 0, b = \pm 2$	$\mathrm{cone}(w_1, w_3, w_6)$, $\mathrm{cone}(w_1, w_3, w_7)$
20	$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|ccc} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & a & b \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right]$ $a = -1,\ -2 \leq b \leq -1$ $or\ (a, b) = (0, -3)$ $or\ a = 1,\ -1 \leq b \leq 0$	$\mathrm{cone}(w_1, w_3, w_6)$, $\mathrm{cone}(w_1, w_3, w_7)$, $\mathrm{cone}(w_2, w_3, w_8)$, $\mathrm{cone}(w_3, w_7, w_8)$
21	$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|ccc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & a & b \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right]$ $(a, b) = (-1, -2)$ $or\ (a, b) = (0, -1)$ $or\ (a, b) = (1, 1)$	$\mathrm{cone}(w_1, w_3, w_6)$, $\mathrm{cone}(w_2, w_3, w_7)$, $\mathrm{cone}(w_3, w_6, w_7)$
22	$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|ccc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & a & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$ $-1 \leq a \leq 2$	$\mathrm{cone}(w_1, w_3, w_6)$, $\mathrm{cone}(w_2, w_3, w_7)$
23	$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|ccc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 2 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$	$\mathrm{cone}(w_1, w_7, w_8)$, $\mathrm{cone}(w_2, w_3, w_7)$, $\mathrm{cone}(w_3, w_6, w_7)$, $\mathrm{cone}(w_3, w_7, w_8)$
24	$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|ccc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 2 & 1 & 1 & 1 & 0 & a & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$ $-4 \leq a \leq 0$	$\mathrm{cone}(w_1, w_3, w_6)$, $\mathrm{cone}(w_2, w_3, w_7)$
26	$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|ccc} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & -2 \end{array}\right]$	$\mathrm{cone}(w_1, w_3, w_7)$, $\mathrm{cone}(w_2, w_6, w_8)$, $\mathrm{cone}(w_3, w_6, w_8)$
31	$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|ccc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$	$\mathrm{cone}(w_1, w_3, w_6)$, $\mathrm{cone}(w_2, w_6, w_8)$, $\mathrm{cone}(w_3, w_6, w_7)$, $\mathrm{cone}(w_3, w_6, w_8)$
32	$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|ccc} 1 & -1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 2 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$	$\mathrm{cone}(w_1, w_7, w_8)$, $\mathrm{cone}(w_2, w_3, w_7)$, $\mathrm{cone}(w_3, w_6, w_7)$, $\mathrm{cone}(w_3, w_7, w_8)$
34	$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|ccc} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & 0 & -1 \\ 0 & -2 & 0 & -2 & -1 & 1 & 1 & 1 \end{array}\right]$	$\mathrm{cone}(w_1, w_3, w_6)$, $\mathrm{cone}(w_1, w_3, w_8)$
35	$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|ccc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & -1 & -1 \\ 0 & -2 & 0 & -2 & -1 & 1 & 1 & 1 \end{array}\right]$	$\mathrm{cone}(w_1, w_3, w_6)$, $\mathrm{cone}(w_1, w_3, w_8)$

Moreover, each of the listed data sets defines a smooth truly almost Fano intrinsic quadric of Picard number three and dimension four.

Chapter four, which was partly presented in [**26**], is devoted to the study of the *base point free monoid*, i.e. the monoid of base point free divisor classes of a Mori dream space X. The first section concerns *embedded monoids*, that means finitely generated monoids in finitely generated abelian groups, and thereby generalizes ideas of the theory on affine semigroups [**18**, Chapter 2] to monoids with non-trivial torsion part. In the subsequent sections, we study the *base point free monoid* of a Mori dream space X, i.e. the embedded monoid of base point free Cartier divisor classes in the Picard group. For complete toric varieties, it is well-known that each semiample Cartier divisor class is base point free, see, for instance, [**21**], i.e. the base point free monoid is saturated. For Mori dream spaces this is in general not true – not even if we restrict to smooth $\mathbb{K}^*$-surfaces, see Example 4.8.4. As a first criterion for the base point free monoid of a Mori dream space to be saturated we show in Corollary 4.3.4 that this is the case if all maximal cones of the minimal toric ambient variety are full-dimensional. For varieties with a torus action of complexity one, we derive the following criterion:

Corollary 4.4.9. *Let X be a rational non-toric projective $\mathbb{Q}$-factorial variety with a torus action of complexity one. If all maximal cones of the minimal toric ambient variety of X are big cones, then the base point free monoid of X is saturated.*

Furthermore, as a consequence of the classifications done in Chapters two and three, we provide sample classes of varieties with saturated base point free monoid, see Corollaries 4.3.6, 4.3.7 and 4.4.13.

Another base point free question was raised by Takao Fujita in the end of the eighties: Fujita's base point free conjecture [**32**] claims that if X is a smooth projective variety with canonical class $\mathcal{K}_X$, then $\mathcal{K}_X + m\mathcal{L}$ is base point free for all $m \geq \dim(X) + 1$ and for all ample Cartier divisor classes $\mathcal{L}$. The study of this conjecture has received much interest; it was proven for toric varieties by Fujino [**30**] and in positive characteristic by Smith [**64**]. Moreover, there are results up to dimension five: For curves, the conjecture is a consequence of Riemann-Roch; for surfaces, it was proven by Reider [**60**]; Ein and Lazarsfeld [**25**] established the proof for threefolds, Kawamata [**46**] proved the conjecture in dimension four and recently, Ye and Zhu [**68**] presented a proof in dimension five. Despite this substantial progress on Fujita's base point free conjecture, it remains in general still widely open. As a consequence of the classifications done in Chapters two and tree, we obtain the following result:

Corollaries 4.3.9, 4.4.14. *Fujita's base point free conjecture is fulfilled if X belongs to one of the following classes of varieties:*

(i) *irreducible smooth rational projective non-toric varieties of Picard number at most two admitting a torus action of complexity one,*
(ii) *smooth intrinsic quadrics of Picard number at most two.*

Moreover, if X is a Mori dream space whose minimal toric ambient variety has only full-dimensional maximal cones, Fujita's base point free conjecture is fulfilled if in addition the anticanoncal class $\mathcal{K}_X$ is semiample or if X has at most log terminal singularities, see Corollary 4.3.8. In Picard number one we use Frobenius numbers to prove the following generalized version for Gorenstein varieties.

Theorem 4.5.5. *Let X be a rational non-toric projective variety with a torus action of complexity one. If $\mathrm{Cl}(X) = \mathbb{Z}$ holds and if X is Gorenstein, then X fulfills Fujita's base point free conjecture.*

A further result for rational non-toric locally factorial varieties with a torus action of complexity one and Picard number two is the following: Proposition 4.6.3 shows that Fujita's base point free conjecture is for these varieties equivalent to the same statement with "base point free" replaced by "semiample". Hence in this case the conjecture is a question of convex geometry rather than of monoid membership. We obtain the following:

Corollary 4.6.4. *Let X be an irreducible normal rational non-toric projective locally factorial variety of Picard number two admitting a torus action of complexity one. If $\mathcal{K}_X$ is semiample or if X is log terminal, then X fulfills Fujita's base point free conjecture.*

In the final part of Chapter four, we present algorithms for the base point free monoid of Mori dream spaces using the combinatorial framework developed in [**3**]. In Section 4.7 we develop algorithms for embedded monoids, among others for computing generators of intersections of embedded monoids and for computing an element of the conductor ideal; see Algorithms 4.7.1, 4.7.3, 4.7.5 and 4.7.7. Applying these algorithms to Mori dream spaces, Section 4.8 provides algorithms for testing whether a given Weil divisor class is base point free and for computing generators of the base point free monoid.

These algorithms, together with the non-emptyness of the conductor ideal of the base point free monoid of a Mori dream space, play an important role in our main algorithm, Algorithm 4.9.4, testing Fujita's base point free conjecture. Since our algorithm makes use of the canonical class $\mathcal{K}_X$, it applies to $\mathbb{Q}$-factorial Mori dream spaces with known canonical class $\mathcal{K}_X$, i.e. for instance if X is spherical or if its Cox ring is a complete intersection, see Remark 4.9.1 for details.

Algorithm 4.9.4. *Input:* A $\mathbb{Q}$-factorial Mori dream space X and its canonical class $\mathcal{K}_X$. *Output:* *True* if X fulfills Fujita's base point free conjecture, *false* if not.

In [**27**], we provide an implementation of our algorithms building on the two `Maple`-based software packages `convex` [**29**] and `MDSpackage` [**38**]. Using this implementation, we prove Fujita's base point free conjecture for a six-dimensional Mori dream space in Example 4.9.5, and in Example 4.9.6, we study a locally factorial variety with a torus action of complexity one that does not fulfill Fujita's base point free conjecture. Note that this depicts an interesting difference to the toric case, where Fujino's proof [**30**] of Fujita's base point free conjecture works also for varieties with arbitrary singularities. A further difference between toric varieties and varieties with a torus action of complexity is illustrated in Example 4.8.4: Here the implementation was applied to construct a first example of a smooth $\mathbb{K}^*$-surface of Picard number twelve admitting a semiample Cartier divisor with base points, thereby illustrating that "semiample" and "base point free" differ in the case of varieties with a torus action of complexity one.

CHAPTER 1

Preliminaries

Throughout this thesis, $\mathbb{K}$ denotes an algebraically closed field of characteristic zero. In Chapter one, we give a short summary of the concepts forming the basis for the subsequent chapters. Note that Chapter one does not contain results of the author of this thesis. Unless stated otherwise, our reference is the book on Cox rings [**3**] written by I. Arzhantsev, U. Derenthal, J. Hausen and A. Laface.

In the first section, Section 1.1, we recall the basic concepts of divisors, Cox rings and good quotients. In Section 1.2, we explain how to construct a variety and in particular all Mori dream spaces starting with some combinatorial data, so called bunched rings. In Section 1.3, we turn to the geometric aspects of Mori dream spaces described in terms of their defining bunched ring. Finally, in Section 1.4, we recall the combinatorial description of rational varieties with a torus action of complexity one via a pair of matrices.

1.1. Divisors, Cox rings and good quotients

We first recall from [**3**] the concepts of divisors and Cox rings and then turn to good quotients which allow us to interpret the Cox sheaf geometrically. Consider an irreducible normal prevariety X over $\mathbb{K}$. A *prime divisor* D *on* X is an irreducible subvariety $D \subseteq X$ of codimension one. The *Weil divisor group* $\mathrm{WDiv}(X)$ is the free abelian group generated by all prime divisors on X. We call its elements, i.e. finite sums $\sum a_D D$ of prime divisors D with integer coefficients a_D, the *Weil divisors* on X. A Weil divisor $D = \sum a_D D$ is called *effective* if $a_D \geq 0$ holds for all prime divisors D; we denote this circumstance by $D \geq 0$. A *principal divisor* is a Weil divisor D admitting a function $f \in \mathbb{K}(X)^*$ such that

$$\mathrm{div}(f) \ = \ \sum \mathrm{ord}_D(f) D$$

holds, where the sum runs over all prime divisors $D \subseteq X$ and $\mathrm{ord}_D(f)$ denotes the vanishing order of f along D. For any open subset $U \subseteq X$, there is a group homomorphism $\mathrm{WDiv}(X) \to \mathrm{WDiv}(U)$ defined by mapping a prime divisor D to its restriction $D_{|U}$, where we set $D_{|U} := 0$ if $D \cap U$ is empty and $D_{|U} := D \cap U$ otherwise. A Weil divisor $D \in \mathrm{WDiv}(X)$ is called a *Cartier divisor* if it is *locally principal*, i.e. if there is an open cover $\{U_i\}_{i \in I}$ such that each $D_{|U_i}$ is principal. By $\mathrm{PDiv}(X) \subseteq \mathrm{CDiv}(X) \subseteq \mathrm{WDiv}(X)$ we denote the subgroups of principal divisors and Cartier divisors in the Weil divisor group. The *divisor class group* and the *Picard group* are the factor groups

$$\mathrm{Cl}(X) \ := \ \mathrm{WDiv}(X)/\,\mathrm{PDiv}(X) \quad \text{and} \quad \mathrm{Pic}(X) \ := \ \mathrm{CDiv}(X)/\,\mathrm{PDiv}(X),$$

respectively. By $[D]$ we denote the class of a Weil divisor D in $\mathrm{Cl}(X)$. Two Weil divisors $D, E \in \mathrm{WDiv}(X)$ are said to be *linearly equivalent* if $[D] = [E]$ holds. The *Picard number* $\varrho(X)$ of X is the rank of its Picard group.

Consider a Weil divisor D on an irreducible, normal prevariety X as well as a non-zero section $f \in \Gamma(X, \mathcal{O}_X(D))$. We call the effective divisor

$$\mathrm{div}_D(f) \ := \ \mathrm{div}(f) + D \ \in \ \mathrm{WDiv}(X)$$

the *D-divisor* of f. To any Weil divisor D on X one associates its *sheaf of $\mathcal{O}_X$-modules* $\mathcal{O}_X(D)$ by setting

$$\Gamma(U, \mathcal{O}_X(D)) \;:=\; \{f \in \mathbb{K}(X)^*;\ \mathrm{div}_D(f)_{|U} \geq 0\} \cup \{0\}$$

for each open subset $U \subseteq X$. Note that $f_1 f_2 \in \Gamma(U, \mathcal{O}_X(D_1 + D_2))$ holds for all $f_i \in \Gamma(U, \mathcal{O}_X(D_i))$, $i = 1, 2$. For a subgroup $K \subseteq \mathrm{WDiv}(X)$ we define the *sheaf of divisorial algebras*

$$\mathcal{S} \;:=\; \bigoplus_{D \in K} S_D\,, \qquad S_D := \mathcal{O}_X(D)\,,$$

where the multiplication in $\mathcal{S}$ is defined by multiplying homogeneous sections in the function field $\mathbb{K}(X)$.

The *complete linear system* $|D|$ of a Weil divisor $D \in \mathrm{WDiv}(X)$ is the set of all effective Weil divisors being linearly equivalent to D, i.e. the set

$$|D| \;:=\; \{E \in \mathrm{WDiv}(X);\ E \geq 0,\, E \sim D\} \;=\; \{\mathrm{div}_D(f);\ f \in \Gamma(X, \mathcal{O}_X(D)) \setminus \{0\}\}.$$

Note that we have a surjection $\mathbb{P}(\Gamma(X, \mathcal{O}_X(D))) \to |D|$ that is a bijection if X is projective. Furthermore, if D and E are linearly equivalent Weil divisors on X, then the complete linear systems $|D|$ and $|E|$ coincide.

Construction 1.1.1. Consider an irreducible, normal prevariety X with finitely generated divisor class group $\mathrm{Cl}(X)$ and only constant invertible global functions, i.e. $\Gamma(X, \mathcal{O}_X) = \mathbb{K}^*$ holds. We fix a subgroup $K \subseteq \mathrm{WDiv}(X)$ such that the map $\pi\colon K \to \mathrm{Cl}(X)$, $D \mapsto [D]$ is surjective. By K_0 we denote the kernel of π. We further choose a group homomorphism $\chi\colon K_0 \to \mathbb{K}(X)^*$ with $\mathrm{div}(\chi(E)) = E$ for all $E \in K_0$. Let $\mathcal{S}$ be the sheaf of divisorial algebras associated with K and denote by $\mathcal{I}$ the sheaf of ideals of $\mathcal{S}$ locally generated by the sections $1 - \chi(E)$, where E runs through all elements of K_0. The *Cox sheaf* of X is the quotient sheaf $\mathcal{R} := \mathcal{S}/\mathcal{I}$ together with the $\mathrm{Cl}(X)$-grading

$$\mathcal{R} \;=\; \bigoplus_{[D] \in \mathrm{Cl}(X)} \mathcal{R}_D, \qquad \mathcal{R}_D \;:=\; p\left(\bigoplus_{D' \in \pi^{-1}([D])} S_{D'}\right),$$

where $p\colon \mathcal{S} \to \mathcal{R}$ denotes the projection. The *Cox ring* of X is the ring of global sections

$$\mathcal{R}(X) \;:=\; \Gamma(X, \mathcal{R}) \;=\; \bigoplus_{[D] \in \mathrm{Cl}(X)} \Gamma(X, \mathcal{R}_{[D]})\,.$$

Note that if $\mathrm{Cl}(X)$ is torsion-free, then the Cox sheaf can be defined in a simpler way by setting $\mathcal{R}_{[D]} := \mathcal{O}_X(D)$. One can show that the Cox ring of X, up to isomorphism, does not depend on the choices made for K and χ.

Example 1.1.2. The Cox ring of the projective space $\mathbb{P}_n$ is $\mathcal{R}(\mathbb{P}_n) = \mathbb{K}[T_0, \ldots, T_n]$, where the grading is given by $\deg(T_i) = 1$ for all $0 \leq i \leq n$.

Definition 1.1.3. Let X be an irreducible normal projective variety with finitely generated divisor class group $\mathrm{Cl}(X)$. If the Cox ring $\mathcal{R}(X)$ of X is finitely generated, then we call X a *Mori dream space*, *MDS* for short.

Definition 1.1.4. Let K be an abelian group and consider a K-graded integral $\mathbb{K}$-algebra $R = \bigoplus_{w \in K} K$.

(i) A non-unit $0 \neq f \in R$ is called *K-prime* if it is K-homogeneous and $f|gh$ with K-homogeneous elements $g, h \in R$ implies $f|g$ or $f|h$.

(ii) We say that R is *K-factorial* or *factorially K-graded* if every non-zero K-homogeneous non-unit $f \in R$ is a product of K-primes.

Theorem 1.1.5. *Let X be an irreducible normal prevariety with only constant invertible global functions and finitely generated divisor class group* $\mathrm{Cl}(X)$. *Then* *the Cox ring $\mathcal{R}(X)$ is integral, normal and* $\mathrm{Cl}(X)$*-factorial. If* $\mathrm{Cl}(X)$ *is torsion-free, then $\mathcal{R}(X)$ is a UFD.*

The aim of the remaining part of this section is to present the geometric interpretation of the Cox sheaf. To do so, we first recall some definitions on algebraic varieties and quasitori. An *(affine) algebraic group* is an (affine) variety G over $\mathbb{K}$ with a group structure such that

$$G \times G \to G,\ (g_1, g_2) \mapsto g_1 g_2 \quad \text{and} \quad G \to G,\ g \mapsto g^{-1}$$

are morphisms of varieties. A *morphism of algebraic groups* G and G' is a homomorphism $G \to G'$ of the underlying groups that is in addition a morphism of varieties. We denote by $\mathbb{K}^*$ the multiplicative group of $\mathbb{K}$. A *character* of an algebraic group G is a morphism of algebraic groups $\chi\colon G \to \mathbb{K}^*$. Together with pointwise multiplication, the characters of an algebraic group G form a group which we denote by $\mathbb{X}(G)$. A *quasitorus* is an affine algebraic group G whose algebra of regular functions $\Gamma(G, \mathcal{O})$ is generated as a $\mathbb{K}$-vector space by the characters $\chi \in \mathbb{X}(G)$. A *torus* is a connected quasitorus. Note that each torus is isomorphic to some $(\mathbb{K}^*)^n$ and that each quasitorus is isomorphic to a direct sum of some finite abelian group and a torus.

Proposition 1.1.6. *There are contravariant functors being essentially inverse to each other between the category of finitely generated abelian groups and the category of quasitori; they are given by*

$$\begin{array}{rcl} K & \mapsto & \mathrm{Spec}(\mathbb{K}[K]), \\ [\bar{\psi}\colon K \to K'] & \mapsto & [\mathrm{Spec}(\mathbb{K}[\bar{\psi}])\colon \mathrm{Spec}(\mathbb{K}[K']) \to \mathrm{Spec}(\mathbb{K}[K])], \\ \mathbb{X}(G) & \mapsfrom & G, \\ [\bar{\varphi}^*\colon \mathbb{X}(G') \to \mathbb{X}(G),\ \chi' \mapsto \chi' \circ \bar{\varphi}] & \mapsfrom & [\bar{\varphi}\colon G \to G']. \end{array}$$

We now recall the correspondence between affine $\mathbb{K}$-algebras graded by a finitely generated group and affine varieties with an action of a quasitorus. Let K be a finitely generated group and let R be a K-graded affine $\mathbb{K}$-algebra. Set $\overline{X} := \mathrm{Spec}(R)$. Choosing K-homogeneous generators $f_1, \ldots, f_r$ of R with $f_i \in R_{w_i}$ gives a closed embedding $\overline{X} \to \mathbb{K}^r,\ x \mapsto (f_1(x), \ldots, f_r(x))$. Note that $\overline{X} \subseteq \mathbb{K}^r$ is invariant under the *diagonal action* of the quasitorus $G := \mathrm{Spec}(\mathbb{K}[K])$ on $\mathbb{K}^r$ given by

$$g \cdot x \ := \ (\chi^{w_1}(g)x_1, \ldots, \chi^{w_r}(g)x_r).$$

Conversely, let G be a quasitorus acting on an affine variety X. We obtain a $\mathbb{X}(G)$-grading of $\Gamma(X, \mathcal{O})$ by setting

$$\Gamma(X, \mathcal{O}) = \bigoplus_{\chi \in \mathbb{X}(G)} \Gamma(X, \mathcal{O})_\chi\,, \quad \Gamma(X, \mathcal{O})_\chi := \{f \in \Gamma(X, \mathcal{O});\ f(g \cdot x) = \chi(g) f(x)\}\,.$$

Proposition 1.1.7. *There are contravariant functors being essentially inverse to each other between the category of affine algebras graded by finitely generated abelian groups and the category of affine varieties with quasitorus action, given by*

$$\begin{array}{rcl} (R, K) & \mapsto & (\mathrm{Spec}(R), \mathrm{Spec}(\mathbb{K}[K])), \\ \big(\psi, \bar{\psi}\big) & \mapsto & \big(\mathrm{Spec}(\psi), \mathrm{Spec}(\mathbb{K}[\bar{\psi}])\big), \\ (\Gamma(X, \mathcal{O}), \mathbb{X}(G)) & \mapsfrom & (X, G), \\ (\varphi^*, \bar{\varphi}^*) & \mapsfrom & (\varphi, \bar{\varphi}). \end{array}$$

An *(affine) G-variety* is an (affine) variety X together with an action $\mu\colon G \times X \to X$ of an algebraic group G such that μ is a morphism. Recall that a *rational representation* of an affine algebraic group G is a morphism $G \to \mathrm{GL}(V)$ of algebraic groups to the affine algebraic group $\mathrm{GL}(V)$ of linear automorphisms of a finite dimensional $\mathbb{K}$-vector space V. A *reductive algebraic group* is an affine algebraic group G such that every rational representation of G splits into irreducible ones. For instance, all finite groups, quasitori and the classical groups $\mathrm{GL}(n)$, $\mathrm{SL}(n)$, $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ are reductive.

Definition 1.1.8. Consider a reductive algebraic group G and a G-variety X. The *ring of invariants* is

$$\mathcal{O}(X)^G \;:=\; \{f \in \Gamma(X, \mathcal{O});\; f(g \cdot x) = f(x) \text{ for all } x \in X,\, g \in G\}\,.$$

A *good quotient* is a morphism $\pi\colon X \to Y$ of varieties such that the following conditions hold:

(i) The morphism π is affine, i.e. the preimage $\pi^{-1}(V)$ of any open affine subset $V \subseteq Y$ is an affine variety.
(ii) The morphism π is G-invariant, i.e. it is constant along orbits.
(iii) The homomorphism of sheaves $\pi^*\colon \mathcal{O}_Y \to (\pi_*\mathcal{O}_X)^G$ is an isomorphism.

A morphism $\pi\colon X \to Y$ of varieties is called *geometric* if it is a good quotient and if each of its fibers consists of one single G-orbit.

Since the quotient space Y of a good quotient $\pi\colon X \to Y$ is unique up to isomorphism, we denote it by $X/\!\!/G$. Note that good quotients for a given variety X need not exist. In case X is an affine G-variety and G is an reductive algebraic group G, Hilbert's Finiteness Theorem ensures that the algebra $\mathcal{O}(X)^G$ is finitely generated. We then obtain a good quotient

$$X \to X/\!\!/G \;=\; \operatorname{Spec}\left(\mathcal{O}(X)^G\right)\,.$$

Example 1.1.9. For any $0 \le i \le n$ we have an action of $G := \mathbb{K}^*$ on $X = \mathbb{K}^n$ via $t \cdot (x_1, \ldots, x_n) := (x_1, \ldots, x_i, tx_{i+1}, \ldots, tx_n)$. Note that $\mathcal{O}(X)^{\mathbb{K}^*} = \mathbb{K}[T_1, \ldots, T_i]$ and $X/\!\!/G = \mathbb{K}^i$ hold.

Now we are ready to present the geometric counterpart of the Cox sheaf $\mathcal{R}$. For this purpose, let X be an irreducible normal variety with only constant invertible global functions and finitely generated divisor class group $\mathrm{Cl}(X)$. If the Cox ring $\mathcal{R}(X)$ of X is finitely generated, then the Cox sheaf $\mathcal{R}$ is locally of finite type allowing us to take the relative spectrum. In this way we obtain an irreducible normal prevariety $\widehat{X} := \operatorname{Spec}_X(\mathcal{R})$. The $\mathrm{Cl}(X)$-grading of the Cox sheaf $\mathcal{R}$ induces an action of $H_X := \operatorname{Spec}(\mathbb{K}[\mathrm{Cl}(X)])$ on $\widehat{X}$. Note that the canonical morphism $p_X\colon \widehat{X} \to X$ is a good quotient for this action and that we have an isomorphism of sheaves $\mathcal{R} \cong (p_X)_*(\mathcal{O}_X)$. Furthermore, there is an open H_X-invariant embedding of $\widehat{X}$ into the affine H_X-variety $\overline{X} := \operatorname{Spec}(\mathcal{R}(X))$ fitting into the following diagram

$$\begin{tikzcd}
\operatorname{Spec}_X(\mathcal{R}) \arrow[r, equal] & \widehat{X} \arrow[r, "\iota"] \arrow[d, "p_X"', "/\!\!/H_X"] & \overline{X} \arrow[r, equal] & \operatorname{Spec}(\mathcal{R}(X)) \\
& X\,. & &
\end{tikzcd}$$

We call $\widehat{X} \to X$ the *characteristic space*, H_X the *characteristic quasitorus* and $\overline{X}$ the *total coordinate space of* X.

Example 1.1.10. For $X = \mathbb{P}_n$, the total coordinate space is $\overline{X} = \mathbb{K}^{n+1}$ and the characteristic space is given by

$$\mathbb{K}^{n+1} \setminus \{0\} \xrightarrow{/\!\!/\mathbb{K}^*} \mathbb{P}_n,\; x \;\mapsto\; [x]\,.$$

1.2. Bunched rings and Mori dream spaces

Similarly to the description of a toric variety in terms of its lattice fan, it is possible to encode Mori dream spaces up to isomorphism in combinatorial objects, so called *bunched rings* [**11, 35**]. The objective of this section is to discuss how to construct a Mori dream space starting from a bunched ring; before doing so, we briefly recall the correspondence between toric varieties and lattice fans.

A *toric variety* is an irreducible normal T-variety X together with a base point $x_0 \in X$ such that T is a torus and such that the orbit map $T \to X$, $t \mapsto t \cdot x_0$ is an open embedding. By a *lattice fan* (N, Σ) we mean a pair consisting of a lattice N and a finite collection Σ of pointed convex polyhedral cones $\sigma \subseteq N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$ with the property that the faces of each $\sigma \in \Sigma$ are contained in Σ and that the intersection of two cones σ, $\sigma' \in \Sigma$ is a face of both σ and σ'. Let us recall the following correspondence between toric varieties and lattice fans:

Proposition 1.2.1. *There are covariant functors being essentially inverse to each other between the category of lattice fans and the category of toric varieties.*

We fix the setting for the rest of the section. Let K be a finitely generated abelian group and R an integral factorially K-graded affine $\mathbb{K}$-algebra. Consider a system $\mathfrak{F} = (f_1, \ldots, f_r)$ of pairwise non-associated K-prime generators of R. The *degree map* is the homomorphism of abelian groups $Q\colon E \to K$ defined by mapping the canonical base vectors $e_i \in E := \mathbb{Z}^r$ to the degrees $w_i := \deg(f_i) \in K$. By $\gamma := \mathbb{Q}^r_{\geq 0}$ we denote the positive orthant. For indices $1 \leq \ell_1 < \ldots < \ell_s \leq r$ we set

$$\gamma_{\ell_1 \ldots \ell_s} \ := \ \gamma_{\ell_1, \ldots, \ell_s} \ := \ \operatorname{cone}(e_{\ell_1}, \ldots, e_{\ell_s}),$$

where we use the notation in the middle in case further clarification is needed. For finitely generated abelian groups B we denote by $B_{\mathbb{Q}}$ the associated rational vector space. We shortly write b for $b \otimes 1 \in B_{\mathbb{Q}}$ and, similarly, we keep the symbols when passing from homomorphisms of groups $B \to B'$ to the linear maps $B_{\mathbb{Q}} \to B'_{\mathbb{Q}}$. The relative interior of a convex polyhedral cone $\sigma \subseteq A_{\mathbb{Q}}$ is denoted by σ°. Consider the canonical base vectors $e_1, \ldots, e_r \in \mathbb{Q}^r$.

Definition 1.2.2. In the above situation, we define the following:

(i) An $\mathfrak{F}$-*face* is a face $\gamma_0 \preceq \gamma$ such that there is some point $x \in \overline{X}$ for which x_i is non-zero if and only if $e_i \in \gamma_0$ holds. We call $Q(\gamma_0)$ a *projected* $\mathfrak{F}$-*face* and denote by $\Omega_{\mathfrak{F}}$ the set of all projected $\mathfrak{F}$-faces.

(ii) An $\mathfrak{F}$-*bunch* is an non-empty subset $\Phi \subseteq \Omega_{\mathfrak{F}}$ such that $\tau_1^{\circ} \cap \tau_2^{\circ} \neq \emptyset$ holds for all $\tau_i \in \Phi$ and such that all $\tilde{\tau} \in \Omega_{\mathfrak{F}}$ with $\tau^{\circ} \subseteq \tilde{\tau}^{\circ}$ for some $\tau \in \Phi$ are contained in Φ.

(iii) An $\mathfrak{F}$-bunch Φ is called *true* if $Q(\gamma_0) \in \Phi$ holds for every facet $\gamma_0 \preceq \gamma$.

(iv) An $\mathfrak{F}$-bunch Φ is called *projective* if there is $u \in K$ such that we have

$$\Phi \ = \ \Phi(u) \ := \ \{\tau \in \Omega_{\mathfrak{F}};\ u \in \tau^{\circ}\}.$$

(v) An $\mathfrak{F}$-bunch is called *maximal* if it cannot be enlarged by adding further projected $\mathfrak{F}$-faces.

(vi) The grading of R is called *almost free* if for every facet $\gamma_0 \preceq \gamma$ the image $Q(\gamma_0 \cap E)$ generates the abelian group K.

Definition 1.2.3. A *bunched ring* is a triple $(R, \mathfrak{F}, \Phi)$, where R is an integral, normal, almost freely factorially K-graded affine $\mathbb{K}$-algebra such that $\mathbb{K}^*$ is the multiplicative group of homogeneous units of R, $\mathfrak{F}$ is a system of pairwise non-associated K-prime generators of R and Φ is a true $\mathfrak{F}$-bunch. We always presume the notation $\mathfrak{F} = (f_1, \ldots, f_r)$.

We now associate a bunched ring $(R, \mathfrak{F}, \Phi)$ with a variety X having R as its Cox ring. Recall that a variety X is called an *A_2-variety* if for each two points $x, x' \in X$ there is an affine, open neighborhood $U \subseteq X$ containing x and x'. We say that a variety X is *A_2-maximal* if it is an A_2-variety and admits no big open embedding $X \subsetneq X'$ into an A_2-variety X', where big means that $X' \setminus X$ is of codimension at least two in X'.

Construction 1.2.4. Let $(R, \mathfrak{F}, K)$ be a bunched ring. An $\mathfrak{F}$-face $\gamma_0 \preceq \gamma$ is called a *relevant face* if $Q(\gamma_0) \in \Phi$ holds. The *collection of relevant faces* and the *covering collection of* Φ are given by

$$\begin{aligned} \mathrm{rlv}(\Phi) \ &:= \ \{\gamma_0 \preceq \gamma;\ \gamma_0 \ \mathfrak{F}\text{-face with } Q(\gamma_0) \in \Phi\}\,, \\ \mathrm{cov}(\Phi) \ &:= \ \{\gamma_0 \in \mathrm{rlv}(\Phi);\ \gamma_0 \text{ minimal with respect to “}\subseteq\text{”}\}\,. \end{aligned}$$

Consider the action of the quasitorus $H := \mathrm{Spec}(\mathbb{K}[K])$ on the affine variety $\overline{X} := \overline{X}(R, \mathfrak{F}, \Phi) := \mathrm{Spec}(R)$. To an $\mathfrak{F}$-face $\gamma_0 \preceq \gamma$ we associate the localization

$$\overline{X}_{\gamma_0} \ := \ \overline{X}_{f_1^{u_1} \cdots f_r^{u_r}} \text{ for some } (u_1, \ldots, u_r) \in \gamma_0^\circ.$$

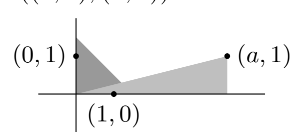
Note that this does not depend on the choice of $(u_1, \ldots, u_r) \in \gamma_0^\circ$. We further set

$$\widehat{X} \ := \ \widehat{X}(R, \mathfrak{F}, \Phi) := \bigcup_{\gamma_0 \in \mathrm{rlv}(\Phi)} \overline{X}_{\gamma_0}\,.$$

The subset $\widehat{X} \subseteq X$ admits a good quotient $p_X \colon \widehat{X} \to X := X(R, \mathfrak{F}, \Phi) := \widehat{X} /\!\!/ H$ and every f_i defines a prime divisor $D_X^i := p_X(V_{\widehat{X}}(f_i))$ on X. We call $X = X(R, \mathfrak{F}, \Phi)$ a *variety arising from a bunched ring*. To simplify the notation, we write $\mathrm{cov}(u)$ and $\mathrm{rlv}(u)$ instead of $\mathrm{cov}(\Phi(u))$ and $\mathrm{rlv}(\Phi(u))$ in case of a projective bunch $\Phi = \Phi(u)$.

Example 1.2.5. The projectivized split vector bundle $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}_1} \oplus \mathcal{O}_{\mathbb{P}_1}(a))$, where $a \in \mathbb{Z}_{\geq 0}$, arises from the following bunched ring $(R, \mathfrak{F}, \Phi)$: the ring $R = \mathbb{K}[T_1, T_2, T_3, T_4]$ is generated by $\mathfrak{F} = (T_1, T_2, T_3, T_4)$; the degrees of the T_i as well as the bunch Φ consisting of the two cones $\mathbb{Q}^2_{\geq 0}$ and $\mathrm{cone}((1,0),(a,1))$ are as follows:

$$(\deg(T_1), \ldots, \deg(T_4)) \ = \ \begin{pmatrix} 1 & 1 & 0 & a \\ 0 & 0 & 1 & 1 \end{pmatrix},$$



Note that X is a toric Hirzebruch surface showing up in the classification of smooth complete toric varieties of Picard number two done by Kleinschmidt [**47**].

Theorem 1.2.6. *Let $X = X(R, \mathfrak{F}, \Phi)$ arise from a bunched ring $(R, \mathfrak{F}, \Phi)$. Then X is an irreducible normal A_2-variety with only constant invertible global functions such that* $\dim(X) = \dim(R) - \dim(K_{\mathbb{Q}})$ *holds. Moreover, $p_X \colon \widehat{X} \to X$ is a characteristic space and we have*

$$\mathrm{Cl}(X) \ \cong \ K, \qquad \mathcal{R}(X) \ \cong \ R\,.$$

Theorem 1.2.7. *Consider an irreducible normal A_2-variety X with only constant invertible global functions, finitely generated divisor class group $K := \mathrm{Cl}(X)$ and finitely generated Cox ring $\mathcal{R}(X)$. Let $\mathfrak{F}$ be any finite system of pairwise nonassociated K-prime generators for R. If X is A_2-maximal, then $X \cong X(R, \mathfrak{F}, \Phi)$ holds with some maximal $\mathfrak{F}$-bunch Φ.*

Later in this thesis, when classifying certain classes of Mori dream spaces, we will relay on the following corollary: it tells us that a Mori dream space is fixed up to isomorphism by its Cox ring and an ample class $u \in \mathrm{Cl}(X)$.

Corollary 1.2.8. *Let X be a Mori dream space. Then $X \cong X(R, \mathfrak{F}, \Phi(u))$ holds with some projective $\mathfrak{F}$-bunch $\Phi(u)$.*

In the rest of the section, we will construct the embedding of a variety $X(R,\mathfrak{F},\Phi)$ into its so called minimal ambient toric variety.

Construction 1.2.9. Consider a bunched ring $(R,\mathfrak{F},\Phi)$ with a system of generators $\mathfrak{F} = (f_1,\dots,f_r)$ and set $E := \mathbb{Z}^r$. With the degree map $Q\colon E \to K$ and with $M := \ker(Q)$, we obtain the following exact sequences of finitely generated abelian groups

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "Q^*"] & F \arrow[r, "P"] & N \end{tikzcd}$$

$$\begin{tikzcd} 0 & K \arrow[l] & E \arrow[l, "Q"'] & M \arrow[l, "P^*"'] & 0, \arrow[l] \end{tikzcd}$$

where P^* is the dual map of P and where we set $L := \ker(P)$. Set $\delta := \gamma^\vee \subseteq F_\mathbb{Q} := F \otimes_\mathbb{Z} \mathbb{Q}$. For each $\gamma_0 \preceq \gamma$ we denote by $\gamma_0^* := \gamma^\perp \cap \delta$ the corresponding face of δ. We define the *envelope* $\mathrm{Env}(\Phi)$ and fans $\widehat{\Sigma}$ and Σ:

$$\begin{aligned} \mathrm{Env}(\Phi) &:= \{\gamma_0 \preceq \gamma;\ \gamma_1 \preceq \gamma_0 \text{ and } Q(\gamma_1)^\circ \subseteq Q(\gamma_0)^\circ \text{ for some } \gamma_1 \in \mathrm{rlv}(\Phi)\}, \\ \widehat{\Sigma} &:= \{\delta_0 \preceq \delta;\ \delta_0 \preceq \gamma_0^* \text{ for some } \gamma_0 \in \mathrm{Env}(\Phi)\}, \\ \Sigma &:= \{P(\gamma_0^*);\ \gamma_0 \in \mathrm{Env}(\Phi)\}. \end{aligned}$$

Consider the action of $H := \mathrm{Spec}(\mathbb{K}[K])$ on $\overline{X} = \overline{X}(R,\mathfrak{F},\Phi)$. By $\overline{Z} := \mathbb{K}^r$, $\widehat{Z}$ and Z we denote the toric varieties associated with the cone δ, the fan $\widehat{\Sigma}$ and the fan Σ, respectively. The system $\mathfrak{F}$ of generators of R defines a closed embedding $\bar{\imath}\colon \overline{X} \to \overline{Z}$, $z \mapsto (f_1(z),\dots,f_r(z))$. Note that $\widehat{Z}$ is an open subset of $\overline{Z}$ that is invariant under the action of H. The toric morphism $p_Z\colon \widehat{Z} \to Z$ corresponding to the map of fans $\widehat{\Sigma} \to \Sigma$ arising from the map of lattices $P\colon F \to N$ is a good quotient and fits into the following commutative diagram

$$\begin{tikzcd} \overline{X} \arrow[r, "\bar{\imath}"] & \overline{Z} \\ \widehat{X} \arrow[u, phantom, "\cup"] \arrow[r, "\hat{\imath}"] \arrow[d, "p_X"'] & \widehat{Z} \arrow[u, phantom, "\cup"] \arrow[d, "p_Z"] \\ X \arrow[r, "\iota"] & Z \end{tikzcd}$$

where $\hat{\imath}$ is the restriction of $\bar{\imath}$ and where we call the induced closed embedding ι of the quotient spaces the *canonical toric embedding* associated with the bunched ring $(R,\mathfrak{F},\Phi)$. Furthermore, we call $Z = Z_\Sigma$ the *minimal ambient toric variety* of X.

1.3. Geometry of Mori dream spaces

Consider a variety X arising from a bunched ring $(R,\mathfrak{F},\Phi)$, for instance a Mori dream space. As we will summarize in this section, many geometric properties of X such as $\mathbb{Q}$-factoriality and smoothness are encoded in the combinatorics of its bunched ring $(R,\mathfrak{F},\Phi)$; for further details see [**3, 11, 35**].

Construction 1.3.1. Consider a variety X arising from a bunched ring $(R,\mathfrak{F},\Phi)$. To any $\mathfrak{F}$-face $\gamma_0 \preceq \gamma$ one associates the locally closed subset

$$\overline{X}(\gamma_0) := \{z \in \overline{X};\ f_i(z) \neq 0 \Leftrightarrow e_i \in \gamma_0 \text{ for all } 1 \leq i \leq r\} \subseteq \overline{X},$$

which we call the *pieces of* $\overline{X}$ associated with γ_0. Note that different $\mathfrak{F}$-faces yield disjoint pieces and that $\overline{X}$ is covered by the pieces of all $\mathfrak{F}$-faces of $(R,\mathfrak{F},\Phi)$. By restricting from $\mathfrak{F}$-faces to relevant faces and by applying the quotient map p_X as defined in Construction 1.2.9, one obtains a decomposition

$$X \;=\; \bigcup_{\gamma_0 \in \mathrm{rlv}(\Phi)} X(\gamma_0)$$

into pairwise disjoint locally closed sets $X(\gamma_0) := p_X(\overline{X}(\gamma_0))$ called the *pieces of* X associated with γ_0.

Proposition 1.3.2. *Consider a variety* $X = X(R, \mathfrak{F}, \Phi)$ *together with the degree map* $Q\colon E \to K$ *and its minimal toric ambient variety* Z. *Inside the class group* $\mathrm{Cl}(X)$, *the Picard group is given by*

$$\mathrm{Pic}(X) \;=\; \mathrm{Pic}(Z) \;=\; \bigcap_{\gamma_0 \in \mathrm{cov}(\Phi)} Q(\gamma_0 \cap E)\,.$$

Let X be an irreducible normal variety. Recall that X is called *locally factorial* if $\mathcal{O}_{X,x}$ is an UFD for each $x \in X$, i.e. if and only if $\mathrm{WDiv}(X) = \mathrm{CDiv}(X)$ holds. Furthermore, we call X $\mathbb{Q}$-factorial if for every Weil divisor some non-zero multiple is Cartier. A variety arising from a bunched ring $(R, \mathfrak{F}, \Phi)$ is called *quasismooth* if $\widehat{X} = \widehat{X}(R, \mathfrak{F}, \Phi)$ is smooth.

Remark 1.3.3. Let $X = X(R, \mathfrak{F}, \Phi)$ be a variety arising from a bunched ring $(R, \mathfrak{F}, \Phi)$ and consider the degree map $Q\colon E \to K$. Then the following statements hold:

(i) The variety X is $\mathbb{Q}$-factorial if and only if $Q(\gamma_0)$ is full-dimensional for each $\gamma_0 \in \mathrm{rlv}(\Phi)$.

(ii) The variety X is locally factorial if and only if Q maps $\mathrm{lin}(\gamma_0) \cap E$ onto $\mathrm{Cl}(X)$ for each $\gamma_0 \in \mathrm{rlv}(\Phi)$. This is exactly the case if its minimal toric ambient variety Z is smooth.

(iii) The variety X is smooth if and only if it is locally factorial and quasismooth.

Definition 1.3.4. Let X be an irreducible normal prevariety and D a Weil divisor on X. The *base locus* and the *stable base locus* of the complete linear system $|D|$ or of the class $w := [D] \in \mathrm{Cl}(X)$ are defined as

$$\mathrm{Bs}|D| \;:=\; \mathrm{Bs}(w) \;:=\; \bigcap_{f \in \Gamma(X, \mathcal{O}_X(D))} \mathrm{Supp}(\mathrm{div}_D(f))\,, \qquad \mathbf{B}(w) \;:=\; \bigcap_{n \in \mathbb{Z}_{\geq 0}} \mathrm{Bs}|nD|\,.$$

An element $x \in \mathrm{Bs}(w)$ is called a *base point* of w. We call $D \in \mathrm{WDiv}(X)$ or its class $w \in \mathrm{Cl}(X)$ *base point free* if the base locus $\mathrm{Bs}(w)$ is empty and *semiample* if its stable base locus is empty. The *effective*, the *semiample* and the *ample cone* are the cones $\mathrm{Eff}(X) \subseteq \mathrm{Cl}(X)_{\mathbb{Q}}$, $\mathrm{SAmple}(X) \subseteq \mathrm{Cl}(X)_{\mathbb{Q}}$ and $\mathrm{Ample}(X) \subseteq \mathrm{Cl}(X)_{\mathbb{Q}}$ generated by the effective, the semiample and the ample Weil divisor classes, respectively. The *moving cone* $\mathrm{Mov}(R) \subseteq K_{\mathbb{Q}}$ of a K-graded affine $\mathbb{K}$-algebra R is the intersection

$$\mathrm{Mov}(R) \;:=\; \bigcap_{i=1}^{r} \mathrm{cone}(w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_r)\,,$$

where $w_1, \ldots, w_r$ denote the degrees of any system of pairwise non-associated homogeneous K-prime generators for R. If $X = X(R, \mathfrak{F}, \Phi)$ holds, then we call $\mathrm{Mov}(R)$ also the moving cone of X and denote it by $\mathrm{Mov}(X)$.

Proposition 1.3.5. *Consider a variety* X *arising from a bunched ring* $(R, \mathfrak{F}, \Phi)$ *together with the degree map* $Q\colon E \to K$ *and its minimal toric ambient variety* Z. *The effective and the moving cone as well as the cones of semiample and ample divisor classes of* X *in* $K_{\mathbb{Q}} = \mathrm{Cl}(X)_{\mathbb{Q}}$ *are given by*

$$\mathrm{Eff}(X) \;=\; Q(\gamma)\,, \qquad \mathrm{Mov}(X) \;=\; \bigcap_{\gamma_0 \preceq \gamma \text{ facet}} Q(\gamma_0)\,,$$

$$\mathrm{SAmple}(X) \;=\; \bigcap_{\gamma_0 \in \mathrm{cov}(\Phi)} Q(\gamma_0)\,, \qquad \mathrm{Ample}(X) \;=\; \bigcap_{\gamma_0 \in \mathrm{cov}(\Phi)} Q(\gamma_0)^\circ\,.$$

Consider a variety X arising from a bunched ring $(R, \mathfrak{F}, \Phi)$. We say that R is a *complete intersection* if the kernel of the epimorphism $\mathbb{K}[T_1, \ldots, T_r] \to R$, $T_i \mapsto f_i$ is generated by K-homogeneous polynomials $g_1, \ldots, g_d$, where $d = r - \dim(R)$ holds.

Proposition 1.3.6. *Let X be a variety arising from a bunched ring $(R, \mathfrak{F}, \Phi)$ that is a complete intersection. In terms of $\mathfrak{F}$ and of the K-homogeneous generators $g_1, \ldots, g_d$ of the kernel of the epimorphism $\mathbb{K}[T_1, \ldots, T_r] \to R$, $T_i \mapsto f_i$, where $d = r - \dim(R)$ holds, the anticanonical class of X is given by*

$$-\mathcal{K}_X \;=\; \sum_{i=1}^{r} \deg(f_i) - \sum_{j=1}^{d} \deg(g_j)\,.$$

Recall that an irreducible normal projective variety is called *Fano* if its anticanonical class $-\mathcal{K}_X$ is ample and *almost Fano* if $-\mathcal{K}_X$ is semiample and big. A *truly almost Fano variety* is a Fano variety being almost Fano but not Fano. If X arises from a bunched ring, the above Propositions show that using the combinatorial data of $(R, \mathfrak{F}, \Phi)$ allows us to compute the semiample and the ample cone of X as well as its anticanonical class $-\mathcal{K}_X$. In this way, we can figure out whether X is (almost) Fano or not. In particular, the K-graded ring R is the Cox ring of a Fano variety if and only if $-\mathcal{K}_X$ belongs to the relative interior of $\mathrm{Mov}(R)$. Recall that the *Fano index* $q(X)$ of a smooth Fano variety X is the largest integer r such that $-\mathcal{K}_X = rw$ holds with some $w \in \mathrm{Cl}(X)$. The *Hilbert series* $H(t)$ of X is

$$H(t) \;:=\; \sum_{n=0}^{\infty} \dim\left(\mathcal{R}(X)_{-n\mathcal{K}_X}\right) t^n\,.$$

We now describe the possible choices of bunches Φ_i for Mori dream spaces sharing the same Cox ring R, i.e. the variation of varieties $X_i = X(R, \mathfrak{F}, \Phi_i)$; it turns out that there are only finitely many choices for Φ_i. By K we denote a finitely generated abelian group and we consider an affine K-graded $\mathbb{K}$-algebra

$$R \;=\; \bigoplus_{w \in K} R_w\,.$$

Then the quasitorus $H := \mathrm{Spec}(\mathbb{K}[K])$ acts on the affine variety $\overline{X} := \mathrm{Spec}(R)$. The *weight cone* $\omega_{\overline{X}}$ of $\overline{X}$ is the cone generated by all $w \in K$ with $R_w \neq \{0\}$ and the *orbit cone* of a point $x \in \overline{X}$ is the cone

$$\omega_x \;:=\; \mathrm{cone}(w \in K;\ f(x) \neq 0 \text{ for some } f \in R_w) \;\subseteq\; K_{\mathbb{Q}} = K \otimes_{\mathbb{Z}} \mathbb{Q}\,.$$

The *set of semistable points* associated with an element $u \in K_{\mathbb{Q}}$ is the H-invariant open subset

$$\overline{X}^{\mathrm{ss}}(u) \;:=\; \{x \in \overline{X};\ f(x) \neq 0 \text{ for some } f \in R_{nu} \text{ with } n > 0\} \;\subseteq\; \overline{X}\,.$$

Note that $\overline{X}^{\mathrm{ss}}(u)$ is non-empty if and only if $u \in \omega_{\overline{X}}$ holds. Fix any system of homogeneous generators $\mathfrak{F} = (f_1, \ldots, f_r)$ for R. Then the set of orbit cones equals the set of projected $\mathfrak{F}$-faces $Q(\gamma_0)$. In particular, there are only finitely many orbit cones. The *GIT-cone* $\lambda(u)$ of an element $u \in \omega_{\overline{X}}$ is the intersection of all orbit cones containing u:

$$\lambda(u) \;:=\; \bigcap_{\substack{x \in \overline{X},\\ u \in \omega_x}} \omega_x \;\subseteq\; K_{\mathbb{Q}}\,.$$

Since there are only finitely many orbit cones, we conclude that there are only finitely many GIT-cones. For every element $u \in K_{\mathbb{Q}}$, the set of semistable points is given in terms of orbit cones and GIT-cones as follows:

$$\overline{X}^{\mathrm{ss}}(u) \ = \ \{x \in X;\ u \in \omega_x\} \ = \ \{x \in X;\ \lambda(u) \subseteq \omega_x\}\,.$$

This shows in particular, that there are only finitely many sets of semistable points. The following theorem tells that the possible sets of semistable points are encoded in a quasifan in $K_{\mathbb{Q}}$.

Theorem 1.3.7. *Set $\overline{X} = \mathrm{Spec}(R)$ and $H := \mathrm{Spec}(\mathbb{K}[K])$ as before. The collection $\Lambda(\overline{X}, H) := \{\lambda(u);\ u \in \omega_{\overline{X}}\}$ of all GIT-cones is a quasifan in $K_{\mathbb{Q}}$ having the weight cone $\omega_{\overline{X}}$ as its support. Moreover, for any two $u_1, u_2 \in \omega_{\overline{X}}$, we have*

$$\lambda(u_1) \subseteq \lambda(u_2) \iff \overline{X}^{ss}(u_1) \supseteq \overline{X}^{ss}(u_2)\,,$$

$$\lambda(u_1) = \lambda(u_2) \iff \overline{X}^{ss}(u_1) = \overline{X}^{ss}(u_2)\,.$$

For a GIT-cone $\lambda \in \Lambda(\overline{X}, H)$, we define the *set of semistable points* as $\overline{X}^{\mathrm{ss}}(\lambda) := \overline{X}^{\mathrm{ss}}(u)$ for any $u \in \lambda^\circ$.

Construction 1.3.8. Consider a finitely generated abelian group K and an integral normal almost freely factorially K-graded affine $\mathbb{K}$-algebra R with $R^* = \mathbb{K}^*$. Each GIT-cone $\lambda \in \Lambda(\overline{X}, H)$ defines a variety $X(\lambda)$ given as quotient space of the good quotient

$$\overline{X}^{\mathrm{ss}}(\lambda) \ \longrightarrow \ X(\lambda) \ := \ \overline{X}^{\mathrm{ss}}(\lambda) /\!\!/ H \ = \ \mathrm{Proj}\left(\bigoplus_{n \in \mathbb{Z}_{\geq 0}} R_{nu} \right)\,,$$

where u is any point in the relative interior of λ. If $R_0 = \mathbb{K}$ holds, then $X(\lambda)$ is projective.

Remark 1.3.9. Consider a finitely generated abelian group K and an integral normal almost freely factorially K-graded normal affine $\mathbb{K}$-algebra R with $R_0 = \mathbb{K}$. Let $\mathfrak{F} = (f_1, \ldots, f_r)$ be a system of pairwise non-associated K-prime generators of R. Each GIT-cone $\lambda \in \Lambda(\overline{X}, H)$ with $\lambda^\circ \subseteq \mathrm{Mov}(R)^\circ$ defines a true projective $\mathfrak{F}$-bunch $\Phi(\lambda) \ := \ \Phi(u)$ for some $u \in \lambda^\circ$. We thus obtain a bunched ring $(R, \mathfrak{F}, \Phi(\lambda))$. In this case, we have $X(\lambda) = X(R, \mathfrak{F}, \Phi(\lambda))$. In particular, all Mori dream spaces with Cox ring $R = \bigoplus_{w \in K} R_w$ are isomorphic to some $X(\lambda)$ with $\lambda \in \Lambda(\overline{X}, H)$ and $\lambda^\circ \subseteq \mathrm{Mov}(R)^\circ$.

A *small quasimodification of* X, *SQM* for short, is a rational map $\varphi\colon X \dashrightarrow X'$ defining an isomorphism between open subsets $U \subseteq X$ and $U' \subseteq X'$ with $X \setminus U$ and $X' \setminus U'$ of codimension at least two in X and X', respectively. We say that a Mori ream space X is *combinatorially minimal* if any birational map $X \dashrightarrow Y$ which is defined in codimension two is a small quasimodification.

Remark 1.3.10. Let X be a variety arising from a bunched ring $(R, \mathfrak{F}, \Phi)$ and consider a GIT-cone $\lambda \in \Lambda(\overline{X}, H)$. Then there is a rational map $\varphi\colon X \dashrightarrow X(\lambda)$. Note that the following holds for φ:

(i) The map φ is birational if and only if $\lambda^\circ \subseteq \mathrm{Eff}(X)^\circ$ holds.
(ii) The map φ is a SQM if and only if $\lambda^\circ \subseteq \mathrm{Mov}(R)^\circ$ holds.
(iii) The map φ is a morphism if and only if $\lambda \subseteq \mathrm{SAmple}(X)$ holds.
(iv) The map φ is an isomorphism if and only if $\lambda^\circ \subseteq \mathrm{Ample}(X)$ holds.

We now recall from **[40, 19]** some basic notation on contractions. Let X be a $\mathbb{Q}$-factorial Mori dream space. A *contraction* is a morphism with connected fibers $\varphi\colon X \to Y$ onto a normal projective variety. Note that there is a bijection between the contractions of X and the faces of $\mathrm{SAmple}(X)$ given by

$$\begin{array}{rcl}\{\text{contractions of } X\} & \longleftrightarrow & \{\text{faces of } \mathrm{SAmple}(X)\}\,,\\ [\varphi\colon X \to Y] & \mapsto & \varphi^*(\mathrm{SAmple}(Y))\,,\\ [\varphi\colon X \to X(\lambda)] & \mapsfrom & \lambda\,.\end{array}$$

We call a contraction $\varphi\colon X \to Y$ *elementary* if $\varrho(X) - \varrho(Y) = 1$ holds, where $\varrho(X)$ and $\varrho(Y)$ denote the Picard numbers of X and Y, respectively. In terms of $\sigma := \varphi^*(\mathrm{SAmple}(Y))$, there are three possibilities for elementary contractions $\varphi\colon X \to Y$:

(i) φ is a *contraction of fiber type*, i.e. σ is contained in the boundary $\partial\mathrm{Eff}(X)$.
(ii) φ is a *birational divisorial contraction*, i.e. $\sigma \subseteq \partial\mathrm{Mov}(X) \setminus \partial\mathrm{Eff}(X)$ holds.
(iii) φ is a *birational small contraction*, i.e. σ is contained in the relative interior of $\mathrm{Mov}(X)$.

As above, write $\overline{X} = \mathrm{Spec}(\mathcal{R}(X))$ and $H_X = \mathrm{Spec}(\mathbb{K}[\mathrm{Cl}(X)])$. In case $\varphi\colon X \to Y$ is birational small, the cone $\sigma = \varphi^*(\mathrm{SAmple}(Y))$ is contained in the relative interior of $\mathrm{Mov}(X)$ and it is furthermore a facet of the semiample cone of X. Thus, there exists a unique $\varrho(X)$-dimensional cone $\lambda' \in \Lambda(\overline{X}, H_X)$ with $\lambda' \subseteq \mathrm{Mov}(X)^\circ$ and $\sigma = \lambda' \cap \mathrm{SAmple}(X)$. The SQM $\psi\colon X \dashrightarrow X(\lambda')$ is the *flip* of φ. By a *rational contraction* of X, we mean a rational map $\varphi\colon X \dashrightarrow Y$ factoring as $X \dashrightarrow X' \to Y$, where $X' \to Y$ is a contraction and where $X \dashrightarrow X'$ is a SQM with a $\mathbb{Q}$-factorial variety X'. Note that there is a bijection between the rational contractions of X and the fan $\mathcal{M}_X := \{\lambda \in \Lambda(\overline{X}, H_X);\ \lambda \subseteq \mathrm{Mov}(X)\}$ given by

$$\begin{array}{rcl}\{\text{rational contractions of } X\} & \longleftrightarrow & \mathcal{M}_X\,,\\ [\varphi\colon X \dashrightarrow Y] & \mapsto & \varphi^*(\mathrm{SAmple}(Y))\,,\\ [\varphi\colon X \dashrightarrow X(\lambda)] & \mapsfrom & \lambda\,.\end{array}$$

1.4. T-varieties of complexity one

Here we recall from **[39, 36, 3]** the Cox ring based approach to irreducible normal projective rational *varieties with a torus action of complexity one*. These varieties, also called *T-varieties of complexity one* for short, are characterized as varieties X admitting an effective action of a torus T of dimension $\dim(X) - 1$. In this section we fix the notation used for these varieties throughout the whole thesis.

Notation 1.4.1. Fix an integer $r \in \mathbb{Z}_{\geq 1}$, a sequence $n_0, \ldots, n_r \in \mathbb{Z}_{\geq 1}$, set $n := n_0 + \ldots + n_r$ and fix integers $m \in \mathbb{Z}_{\geq 0}$ and $0 < s < n + m - r$. A pair (A, P) of *defining matrices* consists of

- a matrix $A := [a_0, \ldots, a_r]$ with pairwise linearly independent column vectors $a_0, \ldots, a_r \in \mathbb{K}^2$,
- a $(r+s)\times(n+m)$-matrix P whose columns are pairwise different primitive vectors generating $\mathbb{Q}^{r+s}$ as a cone and that is of the form

$$P = \begin{bmatrix} L & 0 \\ d & d' \end{bmatrix},$$

where d is an $(s\times n)$-matrix, d' an $(s\times m)$-matrix and L an $(r\times n)$-matrix built from tuples $l_i := (l_{i1}, \ldots, l_{in_i}) \in \mathbb{Z}^{n_i}_{\geq 1}$ as follows

$$L = \begin{bmatrix} -l_0 & l_1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -l_0 & 0 & \ldots & l_r \end{bmatrix}.$$

We denote by v_{ij}, where $0 \leq i \leq r$ and $1 \leq j \leq n_i$ hold, the first n columns of P and by v_k, where $1 \leq k \leq m$ holds, the last m ones. Moreover, $e_{ij}, e_k \in \mathbb{Z}^{n+m}$ are the canonical basis vectors indexed accordingly, i.e. P maps e_{ij} to v_{ij} and e_k to v_k.

Construction 1.4.2. Fix a pair (A, P) as in Notation 1.4.1. Consider the polynomial ring $\mathbb{K}[T_{ij}, S_k]$ in the variables T_{ij}, $0 \le i \le r$, $1 \le j \le n_i$, and S_k, $1 \le k \le m$. For every index $0 \le i \le r$ we define a monomial

$$T_i^{l_i} \ := \ T_{i1}^{l_{i1}} \cdots T_{in_i}^{l_{in_i}} \ \in \ \mathbb{K}[T_{ij}, S_k].$$

Denote by $\mathfrak{I}$ the set of all triples $I = (i_1, i_2, i_3)$ with $0 \le i_1 < i_2 < i_3 \le r$ and define for any $I \in \mathfrak{I}$ a trinomial

$$g_I \ := \ g_{i_1,i_2,i_3} \ := \ \det \begin{bmatrix} T_{i_1}^{l_{i_1}} & T_{i_2}^{l_{i_2}} & T_{i_3}^{l_{i_3}} \\ a_{i_1} & a_{i_2} & a_{i_3} \end{bmatrix}.$$

By P^* we denote the transpose of P. Consider the factor group $K := \mathbb{Z}^{n+m}/\mathrm{im}(P^*)$ and the projection $Q\colon E \to K$, where we set $E := \mathbb{Z}^{n+m}$. We define a K-grading on $\mathbb{K}[T_{ij}, S_k]$ by setting

$$\deg(T_{ij}) \ := \ w_{ij} \ := \ Q(e_{ij}), \qquad \deg(S_k) \ := \ w_k \ := \ Q(e_k).$$

Then the trinomials g_I are K-homogeneous and all of the same degree. In particular, we obtain a K-graded quotient ring

$$R(A, P) \ \ := \ \ \mathbb{K}[T_{ij}, S_k;\ 0 \le i \le r,\ 1 \le j \le n_i, 1 \le k \le m] \ / \ \langle g_I;\ I \in \mathfrak{I} \rangle.$$

Note that the ring $R(A, P)$ is a complete intersection: with $g_i := g_{i,i+1,i+2}$, $0 \le i \le r-2$, we have

$$\langle g_I;\ I \in \mathfrak{I} \rangle \ = \ \langle g_0, \ldots, g_{r-2} \rangle \quad \text{ and } \quad \dim(R(A, P)) \ = \ n + m - (r-1).$$

Remark 1.4.3. The following operations on the columns and rows of the defining matrix P are called *admissible operations* and do not change the isomorphism type of the graded ring $R(A, P)$:

(i) swap two columns inside a block $v_{ij_1}, \ldots, v_{ij_{n_i}}$,
(ii) swap two whole column blocks $v_{ij_1}, \ldots, v_{ij_{n_i}}$ and $v_{i'j_1}, \ldots, v_{i'j_{n_{i'}}}$,
(iii) add integer multiples of the upper r rows to one of the last s rows,
(iv) any elementary row operation among the last s rows,
(v) swap two columns inside the d' block.

The operations of type (iii) and type (iv) do not even change the ring $R(A, P)$, whereas types (i), (ii) and (v) correspond to certain renumberings of the variables of $R(A, P)$ keeping the graded isomorphism type. If we have $n_i = 1$ and $l_{i1} = 1$ in a defining matrix P, then we may eliminate the variable T_{i1} in $R(A, P)$ by modifying P appropriately. This can be repeated until P is *irredundant* in the sense that $l_{i1} + \ldots + l_{in_i} \ge 2$ holds for all $i = 0, \ldots, r$. Hence we can always assume that P is irredundant.

We now construct all irreducible normal projective varieties sharing a given ring $R(A, P)$ as their Cox ring.

Construction 1.4.4. Consider a K-graded ring $R(A, P)$ as in Construction 1.4.2. Then $\mathfrak{F} := \{T_{ij}, S_j\}$ is a system of pairwise non-associated K-prime generators for $R(A, P)$ and any true $\mathfrak{F}$-bunch yields a bunched ring $(R(A, P), \mathfrak{F}, \Phi)$. With Construction 1.2.4, we obtain an irreducible normal A_2-variety X with

$$X \ = \ X(A, P, \Phi) \ := \ X\left(R(A, P), \mathfrak{F}, \Phi\right),$$

$$\dim(X) = s + 1, \qquad \mathrm{Cl}(X) \ \cong \ K, \qquad \text{and} \qquad \mathcal{R}(X) \ \cong \ R(A, P).$$

For an irredundant defining matrix P, the variety $X = X(A, P, \Phi)$ is non-toric if and only if $r \ge 2$ holds. If $\Phi = \Phi(u)$ holds with some $u \in \mathrm{Mov}(R(A, P))^\circ$, we obtain a projective variety $X(A, P, u) := X(A, P, \Phi(u))$.

See [**36**, Theorem 1.5] for the proof that this construction yields indeed all irreducible normal rational projective varieties with a torus action of complexity one.

Construction 1.4.5. Consider a variety $X = X(A, P, \Phi)$ as in Construction 1.4.4 and its minimal ambient toric variety $Z = Z_\Sigma$. Then, with $\lambda := \{0\} \times \mathbb{Q}^s \subseteq \mathbb{Q}^{r+s}$, the canonical basis vectors $e_1, \ldots, e_r \in \mathbb{Z}^{r+s}$ and $e_0 := -e_1 - \ldots - e_r$, the associated *tropical variety* is

$$\operatorname{trop}(X) \;=\; \lambda_0 \cup \ldots \cup \lambda_r \;\subseteq\; \mathbb{Q}^{r+s}, \qquad \text{where} \quad \lambda_i \;:=\; \lambda + \operatorname{cone}(e_i) \quad \text{holds.}$$

Note that for a cone $\sigma \in \Sigma$, there is a face $\gamma_0 \in \operatorname{rlv}(\Phi)$ with $P(\gamma_0^*) = \sigma$ if and only if $\sigma^\circ \cap \operatorname{trop}(X)$ is non-empty.

Definition 1.4.6. Consider a variety $X = X(A, P, \Phi)$ as in Construction 1.4.4 and its minimal ambient toric variety $Z = Z_\Sigma$. A cone $\sigma \in \Sigma$ is called

(i) a *leaf cone* if $\sigma \subseteq \lambda_i$ holds for some $0 \le i \le r$,
(ii) *big* if $\sigma \cap \lambda_i^\circ \neq \emptyset$ holds for each $i = 0, \ldots, r$,
(iii) *elementary big* if it is big, has no rays inside λ and precisely one ray inside λ_i for each $i = 0, \ldots, r$.

We say that the variety X is *weakly tropical* if the fan Σ is supported on the tropical variety $\operatorname{trop}(X)$, i.e. if Σ consists of leaf cones.

CHAPTER 2

Smooth T-varieties of complexity one with Picard number two

A basic intention of chapter two is to contribute to the classification of smooth (almost) Fano varieties with torus action. While smooth toric Fano varieties have already been classified up to dimension nine **[6, 8, 63, 48, 56, 57, 67]** using a description via polytopes, we go one step beyond the toric case and focus on rational varieties with a torus action of complexity one. This means that the general torus orbit is of dimension one less than the variety; see **[65]** for results on smooth Fano threefolds with an action of a two-dimensional torus. The results of this chapter have been published as joint work of the author of this thesis with J. Hausen and M. Nicolussi in **[28]**.

The chapter is organized as follows. In the first section, Section 2.1, we present the classification results. In Section 2.2, we introduce and discuss duplication of free weights and show how to obtain the Fano varieties of Theorem 2.1.2 via this procedure from lower dimensional varieties. Section 2.3 is devoted to the description of the Fano varieties of Theorem 2.1.2 in terms of elementary contractions. As a first step towards the proof of the classification results, Section 2.4 derives constraints on the defining data for smooth X of Picard number two. The final section, Section 2.5, is devoted to proving the main results.

2.1. Classification results in Picard number two

In this section we give an overview on our classification results for smooth rational projective varieties with a torus action of complexity one and Picard number two; the proof is given in Section 2.5. In Theorems 2.1.2 and 2.1.4, we provide a complete list of the smooth projective (almost) Fano varieties. Note that in the setting of irreducible rational projective varieties with a torus action of complexity one, the Cox ring and an ample class fix a variety up to isomorphism.

Theorem 2.1.1. *Every smooth rational irreducible projective non-toric variety of Picard number two that admits a torus action of complexity one is isomorphic to precisely one of the following varieties X, specified by their Cox ring $\mathcal{R}(X)$ and an ample class $u \in \mathrm{Cl}(X)$, where we always have $\mathrm{Cl}(X) = \mathbb{Z}^2$ and the grading is fixed by the matrix $[w_1, \ldots, w_r]$ of generator degrees $\deg(T_i), \deg(S_j) \in \mathrm{Cl}(X)$.*

No.	$\mathcal{R}(X)$	$[w_1,\ldots,w_r]$	u	$\dim(X)$
1	$\frac{\mathbb{K}[T_1,\ldots,T_7]}{\langle T_1T_2T_3^2+T_4T_5+T_6T_7\rangle}$	$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & a & 2-a & b & 2-b \end{bmatrix}$ $1 \le a \le b$	$\begin{bmatrix} 1 \\ 1+b \end{bmatrix}$	4
2	$\frac{\mathbb{K}[T_1,\ldots,T_7]}{\langle T_1T_2T_3+T_4T_5+T_6T_7\rangle}$	$\begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$	4
3	$\frac{\mathbb{K}[T_1,\ldots,T_6]}{\langle T_1T_2T_3^2+T_4T_5+T_6^2\rangle}$	$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 2-a & a & 1 \end{bmatrix}$ $a \ge 1$	$\begin{bmatrix} 1 \\ 1+a \end{bmatrix}$	3

<table>
<tr><td>4</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2^{l_2}+T_3T_4^{l_4}+T_5T_6^{l_6}\rangle}$
$m\geq 0$</td><td>$\left[\begin{array}{cccccc|ccc} 0 & 1 & a & 1 & b & 1 & c_1 & \ldots & c_m \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & \ldots & 1 \end{array}\right]$
$0 \leq a \leq b,\ c_1 \leq \ldots \leq c_m,$
$l_2 = a + l_4 = b + l_6$</td><td>$\begin{bmatrix} d+1 \\ 1 \end{bmatrix}$
$d := \max(b, c_m)$</td><td>$m+3$</td></tr>
<tr><td>5</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2+T_3^2T_4+T_5^2T_6\rangle}$
$m\geq 0$</td><td>$\left[\begin{array}{cccccc|ccc} 0 & 2a+1 & a & 1 & a & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & \ldots & 0 \end{array}\right]$
$a \geq 0$</td><td>$\begin{bmatrix} 2a+2 \\ 1 \end{bmatrix}$</td><td>$m+3$</td></tr>
<tr><td>6</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2T_6\rangle}$
$m\geq 1$</td><td>$\left[\begin{array}{cccccc|ccc} 0 & 2c+1 & a & b & c & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & \ldots & 0 \end{array}\right]$
$a, b, c \geq 0, \quad a < b,$
$a + b = 2c + 1$</td><td>$\begin{bmatrix} 2c+2 \\ 1 \end{bmatrix}$</td><td>$m+3$</td></tr>
<tr><td>7</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$
$m\geq 1$</td><td>$\left[\begin{array}{cccccc|ccc} 0 & 0 & 0 & 0 & -1 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & \ldots & 0 \end{array}\right]$</td><td>$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$</td><td>$m+3$</td></tr>
<tr><td>8</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$
$m\geq 2$</td><td>$\left[\begin{array}{cccccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & a_2 & \ldots & a_m \end{array}\right]$
$0 \leq a_2 \leq \ldots \leq a_m,\ a_m > 0$</td><td>$\begin{bmatrix} 1 \\ a_m+1 \end{bmatrix}$</td><td>$m+3$</td></tr>
<tr><td>9</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$
$m\geq 2$</td><td>$\left[\begin{array}{cccc|ccc} 0 & a_2 & \ldots & a_6 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 & 0 & \ldots & 0 \end{array}\right]$
$0 \leq a_3 \leq a_5 \leq a_6 \leq a_4 \leq a_2,$
$a_2 = a_3 + a_4 = a_5 + a_6$</td><td>$\begin{bmatrix} a_2+1 \\ 1 \end{bmatrix}$</td><td>$m+3$</td></tr>
<tr><td>10</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_5,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$
$m\geq 1$</td><td>$\left[\begin{array}{ccccc|ccc} 1 & 1 & 1 & 1 & 1 & 0 & \ldots & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 & \ldots & 1 \end{array}\right]$</td><td>$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$</td><td>$m+2$</td></tr>
<tr><td>11</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_5,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$
$m\geq 2$</td><td>$\left[\begin{array}{ccccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & a_2 & \ldots & a_m \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & \ldots & 1 \end{array}\right]$
$0 \leq a_2 \leq \ldots \leq a_m,\ a_m > 0$</td><td>$\begin{bmatrix} a_m+1 \\ 1 \end{bmatrix}$</td><td>$m+2$</td></tr>
<tr><td>12</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_5,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$
$m\geq 2$</td><td>$\left[\begin{array}{ccccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 2c & a & b & c & 1 & 1 & \ldots & 1 \end{array}\right]$
$0 \leq a \leq c \leq b,\ a + b = 2c$</td><td>$\begin{bmatrix} 1 \\ 2c+1 \end{bmatrix}$</td><td>$m+2$</td></tr>
<tr><td>13</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\left\langle \begin{array}{c} T_1T_2+T_3T_4+T_5T_6, \\ \lambda T_3T_4+T_5T_6+T_7T_8 \end{array} \right\rangle}$
$\lambda\in\mathbb{K}^*\setminus\{1\}$</td><td>$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$</td><td>$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$</td><td>4</td></tr>
</table>

Moreover, each of the listed data sets defines a smooth rational non-toric projective variety of Picard number two coming with a torus action of complexity one.

Note that by our approach we obtain the Cox ring of the respective varieties for free which in turn allows an explicit treatment of geometric questions by means of Cox ring based techniques. In particular, the canonical divisor of the varieties listed in Theorem 2.1.1 admits a simple description in terms of the defining data. This enables us to determine for every dimension the finitely many (families of) non-toric smooth rational Fano varieties of Picard number two that admit a torus action of complexity one; we refer to Section 2.3 for a geometric description of the listed varieties.

Theorem 2.1.2. *Every smooth rational non-toric Fano variety of Picard number two that admits a torus action of complexity one is isomorphic to precisely one of the following varieties X, specified by their Cox ring $\mathcal{R}(X)$, where the grading by $\mathrm{Cl}(X) = \mathbb{Z}^2$ is given by the matrix $[w_1, \ldots, w_r]$ of generator degrees $\deg(T_i), \deg(S_j) \in \mathrm{Cl}(X)$ and we list the (ample) anticanonical class $-\mathcal{K}_X$.*

No.	$\mathcal{R}(X)$	$[w_1, \ldots, w_r]$	$-\mathcal{K}_X$	$\dim(X)$
1	$\frac{\mathbb{K}[T_1,\ldots,T_7]}{\langle T_1T_2T_3^2+T_4T_5+T_6T_7\rangle}$	$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 3 \\ 4 \end{bmatrix}$	4
2	$\frac{\mathbb{K}[T_1,\ldots,T_7]}{\langle T_1T_2T_3+T_4T_5+T_6T_7\rangle}$	$\begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 4 \end{bmatrix}$	4
3	$\frac{\mathbb{K}[T_1,\ldots,T_6]}{\langle T_1T_2T_3^2+T_4T_5+T_6^2\rangle}$	$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$	3

4.A	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$ $m\geq 0$	$\left[\begin{array}{cccccc\|cccc} 0&1&0&1&0&1&c&0&\dots&0\\ 1&0&1&0&1&0&1&1&\dots&1 \end{array}\right]$ $c \in \{-1,0\}$, $c := 0$ *if* $m=0$	$\begin{bmatrix} 2+c\\ 2+m \end{bmatrix}$	$m+3$
4.B	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2^2+T_3T_4+T_5T_6\rangle}$ $m\geq 0$	$\left[\begin{array}{cccccc\|ccc} 0&1&1&1&1&1&1&\dots&1\\ 1&0&1&0&1&0&1&\dots&1 \end{array}\right]$	$\begin{bmatrix} 3+m\\ 2+m \end{bmatrix}$	$m+3$
4.C	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2^2+T_3T_4^2+T_5T_6^2\rangle}$ $m\geq 0$	$\left[\begin{array}{cccccc\|ccc} 0&1&0&1&0&1&0&\dots&0\\ 1&0&1&0&1&0&1&\dots&1 \end{array}\right]$	$\begin{bmatrix} 1\\ 2+m \end{bmatrix}$	$m+3$
5	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2+T_3^2T_4+T_5^2T_6\rangle}$ $m\geq 1$	$\left[\begin{array}{cccccc\|ccc} 0&2a+1&a&1&a&1&1&\dots&1\\ 1&1&1&0&1&0&0&\dots&0 \end{array}\right]$ $0\leq 2a<m$	$\begin{bmatrix} 2a+m+2\\ 2 \end{bmatrix}$	$m+3$
6	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2T_6\rangle}$ $m\geq 2$	$\left[\begin{array}{cccccc\|ccc} 0&2c+1&a&b&c&1&1&\dots&1\\ 1&1&1&1&1&0&0&\dots&0 \end{array}\right]$ $a,b,c\geq 0,\quad a<b,$ $a+b=2c+1,$ $m>3c+1$	$\begin{bmatrix} 3c+2+m\\ 3 \end{bmatrix}$	$m+3$
7	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$ $1\leq m\leq 3$	$\left[\begin{array}{cccccc\|ccc} 0&0&0&0&-1&1&1&\dots&1\\ 1&1&1&1&1&1&0&\dots&0 \end{array}\right]$	$\begin{bmatrix} m\\ 4 \end{bmatrix}$	$m+3$
8	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$ $m\geq 2$	$\left[\begin{array}{cccccc\|cccc} 0&0&0&0&0&0&1&1&\dots&1\\ 1&1&1&1&1&1&0&a_2&\dots&a_m \end{array}\right]$ $0\leq a_2\leq\dots\leq a_m,$ $a_m\in\{1,2,3\},$ $4+\sum_{k=2}^m a_k>ma_m$	$\begin{bmatrix} m\\ 4+\sum_{k=2}^m a_k \end{bmatrix}$	$m+3$
9	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$ $m\geq 2$	$\left[\begin{array}{cccc\|ccc} 0&a_2&\dots&a_6&1&\dots&1\\ 1&1&\dots&1&0&\dots&0 \end{array}\right]$ $0\leq a_3\leq a_5\leq a_6\leq a_4\leq a_2,$ $a_2=a_3+a_4=a_5+a_6,$ $2a_2<m$	$\begin{bmatrix} 2a_2+m\\ 4 \end{bmatrix}$	$m+3$
10	$\frac{\mathbb{K}[T_1,\dots,T_5,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$ $1\leq m\leq 2$	$\left[\begin{array}{ccccc\|ccc} 1&1&1&1&1&0&\dots&0\\ -1&1&0&0&0&1&\dots&1 \end{array}\right]$	$\begin{bmatrix} 3\\ m \end{bmatrix}$	$m+2$
11	$\frac{\mathbb{K}[T_1,\dots,T_5,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$ $m\geq 2$	$\left[\begin{array}{ccccc\|cccc} 1&1&1&1&1&0&a_2&\dots&a_m\\ 0&0&0&0&0&1&1&\dots&1 \end{array}\right]$ $0\leq a_2\leq\dots\leq a_m,$ $a_m\in\{1,2\},$ $3+\sum_{k=2}^m a_k>ma_m$	$\begin{bmatrix} 3+\sum_{k=2}^m a_k\\ m \end{bmatrix}$	$m+2$
12	$\frac{\mathbb{K}[T_1,\dots,T_5,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$ $m\geq 2$	$\left[\begin{array}{ccccc\|cccc} 1&1&1&1&1&0&0&\dots&0\\ 0&2c&a&b&c&1&1&\dots&1 \end{array}\right]$ $0\leq a\leq c\leq b,\ a+b=2c,$ $3c<m$	$\begin{bmatrix} 3\\ 3c+m \end{bmatrix}$	$m+2$
13	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\left\langle \begin{array}{c} T_1T_2+T_3T_4+T_5T_6,\\ \lambda T_3T_4+T_5T_6+T_7T_8 \end{array}\right\rangle}$ $\lambda\in\mathbb{K}^*\setminus\{1\}$	$\begin{bmatrix} 1&0&1&0&1&0&1&0\\ 0&1&0&1&0&1&0&1 \end{bmatrix}$	$\begin{bmatrix} 2\\ 2 \end{bmatrix}$	4

Moreover, each of the listed data sets defines a smooth rational non-toric Fano variety of Picard number two coming with a torus action of complexity one.

For $\mathbb{K}=\mathbb{C}$, the assumption of rationality can be omitted in Theorem 2.1.2 due to [**45**, Sec. 2.1] and [**3**, Rem. 4.4.1.5]. A closer look to the varieties of Theorem 2.1.2 reveals that they all are obtained from a series of lower dimensional varieties via iterating the following procedure: we take a certain $\mathbb{P}_1$-bundle over the given variety, apply a natural series of flips and then contract a prime divisor. In terms of Cox rings, this generalized cone construction simply means duplicating a free weight, i.e. given a variable not showing up in the defining relations, one adds a further one of the same degree, see Section 2.2. Proposition 2.2.4 and Theorem 2.2.5 then yield the following.

Corollary 2.1.3. *Every smooth rational non-toric Fano variety with a torus action of complexity one and Picard number two arises via iterated duplication of a free weight from a smooth rational projective (not necessarily Fano) variety with a torus action of complexity one, Picard number two and dimension at most seven.*

Note that we cannot expect such a statement in general: Remark 2.2.7 shows that the smooth toric Fano varieties of Picard number two do not allow a bound d such that they all arise via iterated duplication of free weights from smooth varieties of dimension at most d.

Similar to the Fano varieties, we can figure out the almost Fano varieties from Theorem 2.1.1, i.e. those with a big and nef anticanonical divisor. In general, i.e. without the assumption of a torus action, the classification of smooth almost Fano varieties of Picard number two is widely open; for the threefold case, we refer to the work of Jahnke, Peternell and Radloff [**42, 43**]. In the setting of a torus action of complexity one, the following result together with Theorem 2.1.2 settles the problem in any dimension; by a *truly almost Fano variety* we mean an almost Fano variety which is not Fano.

Theorem 2.1.4. *Every smooth rational non-toric truly almost Fano variety of Picard number two that admits a torus action of complexity one is isomorphic to precisely one of the following varieties X, specified by their Cox ring $\mathcal{R}(X)$ and an ample class $u \in \mathrm{Cl}(X)$, where we always have $\mathrm{Cl}(X) = \mathbb{Z}^2$ and the grading is fixed by the matrix $[w_1, \ldots, w_r]$ of generator degrees $\deg(T_i), \deg(S_j) \in \mathrm{Cl}(X)$.*

<table>
<tr><th>No.</th><th>$\mathcal{R}(X)$</th><th>$[w_1, \ldots, w_r]$</th><th>u</th><th>$\dim(X)$</th></tr>
<tr><td>4.A</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$
$m\geq 1$</td><td>$\left[\begin{array}{cccccc|ccc} 0 & 1 & 0 & 1 & 0 & 1 & c_1 & \ldots & c_m \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & \ldots & 1 \end{array}\right]$
$c_1 \leq \ldots \leq c_m$
$d := \max(0, c_m)$
$(2+m)d = 2 + c_1 + \cdots + c_m$</td><td>$\begin{bmatrix} 1 \\ 1+d \end{bmatrix}$</td><td>$m+3$</td></tr>
<tr><td>4.B</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2^2+T_3T_4+T_5T_6\rangle}$
$m\geq 1$</td><td>$\left[\begin{array}{cccccc|cccc} 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & \ldots & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & \ldots & 1 \end{array}\right]$</td><td>$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$</td><td>$m+3$</td></tr>
<tr><td>4.C</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2^2+T_3T_4^2+T_5T_6^2\rangle}$
$m\geq 1$</td><td>$\left[\begin{array}{cccccc|cccc} 0 & 1 & 0 & 1 & 0 & 1 & -1 & 0 & \ldots & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & \ldots & 1 \end{array}\right]$</td><td>$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$</td><td>$m+3$</td></tr>
<tr><td>4.D</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2^2+T_3T_4^2+T_5T_6\rangle}$
$m\geq 0$</td><td>$\left[\begin{array}{cccccc|ccc} 0 & 1 & 0 & 1 & 1 & 1 & 1 & \ldots & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & \ldots & 1 \end{array}\right]$</td><td>$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$</td><td>$m+3$</td></tr>
<tr><td>4.E</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2^3+T_3T_4+T_5T_6\rangle}$
$m\geq 0$</td><td>$\left[\begin{array}{cccccc|ccc} 0 & 1 & 2 & 1 & 2 & 1 & 2 & \ldots & 2 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & \ldots & 1 \end{array}\right]$</td><td>$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$</td><td>$m+3$</td></tr>
<tr><td>4.F</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2^3+T_3T_4^2+T_5T_6^2\rangle}$
$m\geq 0$</td><td>$\left[\begin{array}{cccccc|ccc} 0 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & \ldots & 1 \end{array}\right]$</td><td>$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$</td><td>$m+3$</td></tr>
<tr><td>5</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2+T_3^2T_4+T_5^2T_6\rangle}$
$m\geq 0$</td><td>$\left[\begin{array}{cccccc|ccc} 0 & 2a+1 & a & 1 & a & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & \ldots & 0 \end{array}\right]$
$m = 2a$</td><td>$\begin{bmatrix} m+2 \\ 1 \end{bmatrix}$</td><td>$m+3$</td></tr>
<tr><td>6</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2T_6\rangle}$
$m\geq 1$</td><td>$\left[\begin{array}{cccccc|ccc} 0 & 2c+1 & a & b & c & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & \ldots & 0 \end{array}\right]$
$a, b, c \geq 0, \quad a < b,$
$a + b = 2c + 1,$
$m = 3c + 1$</td><td>$\begin{bmatrix} 2c+2 \\ 1 \end{bmatrix}$</td><td>$m+3$</td></tr>
<tr><td>7</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$
$m=4$</td><td>$\left[\begin{array}{cccccc|cccc} 0 & 0 & 0 & 0 & -1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{array}\right]$</td><td>$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$</td><td>7</td></tr>
<tr><td>8</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$
$m\geq 2$</td><td>$\left[\begin{array}{cccccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & a_2 & \ldots & a_m \end{array}\right]$
$0 \leq a_2 \leq \ldots \leq a_m, \ a_m > 0,$
$4 + a_2 + \ldots + a_m = m a_m$</td><td>$\begin{bmatrix} 1 \\ a_m + 1 \end{bmatrix}$</td><td>$m+3$</td></tr>
<tr><td>9</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$
$m\geq 2$</td><td>$\left[\begin{array}{cccc|ccc} 0 & a_2 & \ldots & a_6 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 & 0 & \ldots & 0 \end{array}\right]$
$0 \leq a_3 \leq a_5 \leq a_6 \leq a_4 \leq a_2,$
$a_2 = a_3 + a_4 = a_5 + a_6,$
$m = 2a_2$</td><td>$\begin{bmatrix} a_2 + 1 \\ 1 \end{bmatrix}$</td><td>$m+3$</td></tr>
<tr><td>10</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_5,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$
$m=3$</td><td>$\left[\begin{array}{ccccc|ccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right]$</td><td>$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$</td><td>5</td></tr>
</table>

11	$\frac{\mathbb{K}[T_1,\ldots,T_5,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$ $m\geq 2$	$\left[\begin{array}{ccccc\|cccc} 1&1&1&1&1&0&a_2&\ldots&a_m\\ 0&0&0&0&0&1&1&\ldots&1\end{array}\right]$ $0\leq a_2\leq\ldots\leq a_m,\ a_m>0,$ $3+a_2+\ldots+a_m=ma_m$	$\begin{bmatrix}1\\ a_m+1\end{bmatrix}$	$m+2$
12	$\frac{\mathbb{K}[T_1,\ldots,T_5,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$ $m\geq 3$	$\left[\begin{array}{ccccc\|cccc} 1&1&1&1&1&0&0&\ldots&0\\ 0&2c&a&b&c&1&1&\ldots&1\end{array}\right]$ $0\leq a\leq c\leq b,\ a+b=2c,$ $m=3c$	$\begin{bmatrix}1\\ 2c+1\end{bmatrix}$	$m+2$

Moreover, each of the listed data sets defines a smooth rational non-toric truly almost Fano variety of Picard number two coming with a torus action of complexity one.

2.2. Duplicating free weights

As mentioned in the introduction, there are by a result of Liendo and Süß [**49**, Thm. 6.5] up to isomorphism just two smooth non-toric projective varieties with a torus action of complexity one and Picard number one, namely the smooth projective quadrics in dimensions three and four. In Picard number two we obtained examples in every dimension and this even holds when we restrict to the Fano case. Nevertheless, also in Picard number two we observe a certain finiteness feature: each Fano variety listed in Theorem 2.1.2 arises from a smooth, but not necessarily Fano, variety of dimension at most seven via *duplicating free weights*.

For the precise treatment, the setting of bunched rings $(R, \mathfrak{F}, \Phi)$ is most appropriate. Recall from Section 1.2 that R is an integral normal almost freely factorially K-graded $\mathbb{K}$-algebra, $\mathfrak{F}$ a system of pairwise non-associated K-prime generators for R and Φ a certain collection of polyhedral cones in $K_{\mathbb{Q}}$ defining an open set $\widehat{X} \subseteq \overline{X} = \operatorname{Spec} R$ with a good quotient $X(R, \mathfrak{F}, \Phi) := \widehat{X} /\!\!/ H$ by the action of the quasitorus $H = \operatorname{Spec} \mathbb{K}[K]$ on $\overline{X}$. Recall that $X := X(R, \mathfrak{F}, \Phi)$ is called a variety arising from a bunched ring. Dimension, divisor class group and Cox ring of X are given by

$$\dim(X) \;=\; \dim(R) - \dim(K_{\mathbb{Q}}), \qquad \operatorname{Cl}(X) \;=\; K, \qquad \mathcal{R}(X) \;=\; R.$$

Construction 2.2.1. Let $R = \mathbb{K}[T_1, \ldots, T_r]/\langle g_1, \ldots, g_s\rangle$ be a K-graded algebra presented by K-homogeneous generators T_i and relations $g_j \in \mathbb{K}[T_1, \ldots, T_{r-1}]$. By *duplicating the free weight* $\deg(T_r)$ we mean passing from R to the K-graded algebra

$$R' \;:=\; \mathbb{K}[T_1, \ldots, T_r, T_{r+1}]/\langle g_1, \ldots, g_s\rangle, \qquad \deg(T_{r+1}) \;:=\; \deg(T_r) \;\in\; K,$$

where $g_j \in \mathbb{K}[T_1, \ldots, T_{r-1}] \subseteq \mathbb{K}[T_1, \ldots, T_r, T_{r+1}]$. If in this situation $(R, \mathfrak{F}, \Phi)$ is a bunched ring with $\mathfrak{F} = (T_1, \ldots, T_r)$, then $(R', \mathfrak{F}', \Phi)$ is a bunched ring with $\mathfrak{F}' = (T_1, \ldots, T_r, T_{r+1})$.

Proof. The $\mathbb{K}$-algebra R' is integral and normal and, by [**9**, Thm. 1.4], factorially K-graded. Obviously, the K-grading is almost free in the sense of [**3**, Def. 3.2.1.1]. Moreover, $(R, \mathfrak{F})$ and $(R', \mathfrak{F}')$ have the same sets of generator weights in the common grading group K and the collection of projected $\mathfrak{F}'$-faces equals the collection of projected $\mathfrak{F}$-faces. We conclude that Φ is a true $\mathfrak{F}'$-bunch in the sense of [**3**, Def. 3.2.1.1] and thus $(R', \mathfrak{F}', \Phi)$ is a bunched ring. $\square$

The word "free" in Construction 2.2.1 indicates that the variable T_r does not occur in the relations g_j. Here are the basic features of the procedure.

Proposition 2.2.2. *Let* $(R', \mathfrak{F}', \Phi)$ *arise from the bunched ring* $(R, \mathfrak{F}, \Phi)$ *via Construction 2.2.1. Set* $X' := X(R', \mathfrak{F}', \Phi)$ *and* $X := X(R, \mathfrak{F}, \Phi)$.

(i) *We have* $\dim(X') = \dim(X) + 1$.
(ii) *The cones of semiample divisor classes satisfy* $\operatorname{SAmple}(X') = \operatorname{SAmple}(X)$.

(iii) *The variety X' is smooth if and only if X is smooth.*
(iv) *The ring R' is a c.i. if and only if R is a c.i..*
(v) *If R is a c.i.,* $\deg(T_r)$ *semiample and X Fano, then X' is Fano.*

Proof. By construction, $\dim(R') = \dim(R) + 1$ holds. Since R and R' have the same grading group K, we obtain (i). Moreover, R and R' have the same defining relations g_j, hence we have (iv). According to [**3**, Prop. 3.3.2.9], the semiample cone is the intersection of all elements of Φ and thus (ii) holds.

To obtain the third assertion, we show first that $\widehat{X}'$ is smooth if and only if $\widehat{X}$ is smooth. For every relevant $\mathfrak{F}$-face $\gamma_0 \preceq \mathbb{Q}^r_{\geq 0}$ consider

$$\gamma_0' \;:=\; \gamma_0 + \mathrm{cone}(e_{r+1}), \qquad \gamma_0'' \;:=\; \mathrm{cone}(e_i;\ 1 \leq i < r,\ e_i \in \gamma_0) + \mathrm{cone}(e_{r+1}).$$

Then $\gamma_0, \gamma_0', \gamma_0'' \preceq \mathbb{Q}^{r+1}_{\geq 0}$ are relevant $\mathfrak{F}'$-faces and, in fact, all relevant $\mathfrak{F}'$-faces are of this form. Since the variables T_r and T_{r+1} do not appear in the relations g_j, we see that a piece $\overline{X}(\gamma_0)$ is smooth if and only if the pieces $\overline{X}'(\gamma_0)$, $\overline{X}'(\gamma_0')$ and $\overline{X}'(\gamma_0'')$ are smooth. Now [**3**, Cor. 3.3.1.11] gives (iii).

Finally, we show (v). As we have complete intersection Cox rings, [**3**, Prop. 3.3.3.2] applies and we obtain

$$-\mathcal{K}_{X'} \;=\; \sum_{i=1}^{r+1} \deg(T_i) - \sum_{j=1}^{s} \deg(g_j) \;=\; -\mathcal{K}_X + \deg(T_{r+1}).$$

Since X and X' share the same ample cone, we conclude that ampleness of $-\mathcal{K}_X$ implies ampleness of $-\mathcal{K}_{X'}$. $\square$

We interpret the duplication of free weights in terms of birational geometry: it turns out to be a composition of a contraction of fiber type, a series of flips and a divisorial contraction, where all contractions are elementary, i.e. of relative Picard number one; see [**19**] for a detailed study of the latter type of maps in the context of general smooth Fano 4-folds.

Proposition 2.2.3. *Let $(R', \mathfrak{F}', \Phi)$ arise from the bunched ring $(R, \mathfrak{F}, \Phi)$ via Construction 2.2.1. Set $X' := X(R', \mathfrak{F}', \Phi)$ and $X := X(R, \mathfrak{F}, \Phi)$. Assume that X is $\mathbb{Q}$-factorial. Then there is a sequence*

$$X \;\longleftarrow\; \widetilde{X}_1 \;\dashrightarrow\; \ldots \;\dashrightarrow\; \widetilde{X}_t \;\longrightarrow\; X',$$

where $\widetilde{X}_1 \to X$ is a contraction of fiber type with fibers $\mathbb{P}_1$, every $\widetilde{X}_i \dashrightarrow \widetilde{X}_{i+1}$ is a flip and $\widetilde{X}_t \to X'$ is the contraction of a prime divisor. If $\deg(T_r) \in K$ *is Cartier, then $\widetilde{X}_1 \to X$ is the $\mathbb{P}_1$-bundle associated with the divisor on X corresponding to T_r.*

Proof. In order to define $\widetilde{X}_1$, we consider the canonical toric embedding $X \subseteq Z$ in the sense of [**3**, Constr. 3.2.5.3]. Let Σ be the fan of Z and $P = [v_1, \ldots, v_r]$ be the matrix having the primitive generators $v_i \in \mathbb{Z}^n$ of the rays of Σ as its columns. Define a further matrix

$$\widetilde{P} \;:=\; \begin{bmatrix} v_1 & \ldots & v_{r-1} & v_r & 0 & 0 \\ 0 & \ldots & 0 & -1 & 1 & -1 \end{bmatrix}.$$

We denote the columns of $\widetilde{P}$ by $\widetilde{v}_1, \ldots, \widetilde{v}_r, \widetilde{v}_+, \widetilde{v}_- \in \mathbb{Z}^{n+1}$, write ϱ_+, ϱ_- for the rays through $\widetilde{v}_+$, $\widetilde{v}_-$ and define a fan

$$\widetilde{\Sigma}_1 \;:=\; \{\widetilde{\sigma} + \varrho_+,\, \widetilde{\sigma} + \varrho_-,\, \widetilde{\sigma};\ \sigma \in \Sigma\}, \qquad\qquad \widetilde{\sigma} \;:=\; \mathrm{cone}(\widetilde{v}_i; v_i \in \sigma).$$

The projection $\mathbb{Z}^{n+1} \to \mathbb{Z}^n$ is a map of fans $\widetilde{\Sigma}_1 \to \Sigma$. The associated toric morphism $\widetilde{Z}_1 \to Z$ has fibers $\mathbb{P}_1$. If the toric divisor D_r corresponding to the ray through v_r is Cartier, then $\widetilde{Z}_1 \to Z$ is the $\mathbb{P}_1$-bundle associated with D_r. We define $\widetilde{X}_1 \subseteq \widetilde{Z}_1$ to be the preimage of $X \subseteq Z$. Then $\widetilde{X}_1 \to X$ has fibers $\mathbb{P}_1$. If $\deg(T_r)$ is Cartier, then so is D_r and hence $\widetilde{X}_1 \to X$ inherits the $\mathbb{P}_1$-bundle structure.

Now we determine the Cox ring of the variety $\widetilde{X}_1$. For this, observe that the projection $\mathbb{Z}^{r+2} \to \mathbb{Z}^r$ defines a lift of $\widetilde{Z}_1 \to Z$ to the toric characteristic spaces and thus leads to the commutative diagram

$$\begin{tikzcd}
\widetilde{\pi}^\sharp(\widetilde{X}_1) \arrow[d, "\widetilde{\pi}"'] & \subseteq & \widetilde{W}_1 \arrow[r] \arrow[d, "\widetilde{\pi}"'] & W \arrow[d, "\pi"] & \supseteq & \pi^\sharp(X) \arrow[d, "\pi"] \\
\widetilde{X}_1 & \subseteq & \widetilde{Z}_1 \arrow[r] & Z & \supseteq & X
\end{tikzcd}$$

where $\widetilde{\pi}^\sharp(\widetilde{X}_1)$ and $\pi^\sharp(X)$ denote the proper transforms with respect to the downwards toric morphisms. Pulling back the defining equations of $\pi^\sharp(X) \subseteq W$, we see that $\widetilde{\pi}^\sharp(\widetilde{X}_1) \subseteq \widetilde{W}_1$ has coordinate algebra $\widetilde{R} := R[S^+, S^-]$ graded by $\widetilde{K} := K \times \mathbb{Z}$ via

$$\deg(T_i) := (w_i, 0), \qquad w^+ := \deg(S^+) := (w_r, 1), \qquad w^- := \deg(S^-) := (0, 1),$$

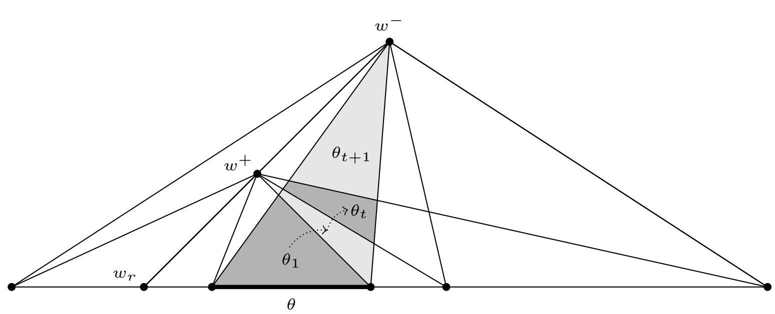
where $w_i := \deg(T_i) \in K$ holds. The $\mathbb{K}$-algebra $\widetilde{R}$ is integral and normal and, by [**9**, Thm. 1.4], factorially $\widetilde{K}$-graded. Moreover the $\widetilde{K}$-grading is almost free, as the K-grading of R has this property and $\widetilde{\mathfrak{F}} = (T_1, \ldots, T_r, S^+, S^-)$ is a system of pairwise non-associated $\widetilde{K}$-prime generators. We conclude that $\widetilde{R}$ is the Cox ring of $\widetilde{X}_1$.

Next we look for the defining bunch of cones for $\widetilde{X}_1$. Observe that K sits inside $\widetilde{K}$ as $K \times \{0\}$. With $\theta := \mathrm{SAmple}(X) \times \{0\}$ we obtain a GIT-cone $\theta_1 := \mathrm{cone}(\theta, w^+) \cap \mathrm{cone}(\theta, w^-)$ of the $\widetilde{K}$-graded ring $\widetilde{R}$. The associated bunch $\widetilde{\Phi}_1$ consists of all cones of the form

$$\widetilde{\tau} + \mathrm{cone}(w^+), \qquad \widetilde{\tau} + \mathrm{cone}(w^-), \qquad \widetilde{\tau} + \mathrm{cone}(w^+, w^-),$$

where $\widetilde{\tau} = \tau \times \{0\}$, $\tau \in \Phi$. Since Φ is a true bunch, so is $\widetilde{\Phi}_1$. Together we obtain a bunched ring $(\widetilde{R}, \widetilde{\mathfrak{F}}, \widetilde{\Phi}_1)$. By construction, the fan corresponding to $\widetilde{\Phi}_1$ via Gale duality is $\widetilde{\Sigma}_1$. We conclude that $\widetilde{X}_1$ is the variety associated with $(\widetilde{R}, \widetilde{F}, \widetilde{\Phi}_1)$ and $\widetilde{X}_1 \subseteq \widetilde{Z}_1$ is the canonical toric embedding.

Observe that $\widetilde{X}_1 \to X$ corresponds to the passage from the GIT-cone θ_1 to the facet θ. In particular, we see that $\widetilde{X}_1 \to X$ is an elementary contraction of fiber type. To obtain the flips and the final divisorial contraction, we consider the full GIT-fan.



Important are the GIT-cones inside $\theta + \mathrm{cone}(w^-)$. There we have the facet θ and the semiample cone θ_1 of $\widetilde{X}_1$. Proceeding in the direction of w^-, we come across other full-dimensional GIT-cones, say $\theta_2, \ldots, \theta_{t+1}$. This gives a sequence of flips $\widetilde{X}_1 \dashrightarrow \ldots \dashrightarrow \widetilde{X}_t$, where $\widetilde{X}_i$ is the variety with semiample cone θ_i. Passing from θ_t to θ_{t+1} gives a morphism $\widetilde{X}_t \to \widetilde{X}_{t+1}$ contracting the prime divisor corresponding to the variable S^- of the Cox ring $\widetilde{R}$ of $\widetilde{X}_t$. Note that $\widetilde{X}_{t+1}$ is $\mathbb{Q}$-factorial, as it is the GIT-quotient associated with a full-dimensional chamber.

We show $\widetilde{X}_{t+1} \cong X'$. Recall that X' arises from X by duplicating the weight $\deg(T_r)$. We have $\mathrm{Cl}(X') = K$ and the Cox ring $R' = R[T_{r+1}]$ of X' is K-graded via $\deg(T_i) = w_i$ for $i = 1, \ldots, r$ and $\deg(T_{r+1}) = w_r$. In particular, the fan of the canonical toric ambient variety of X' has as its primitive ray generators the columns of the matrix

$$P' \ = \ \begin{bmatrix} v_1 & \ldots & v_{r-1} & v_r & 0 \\ 0 & \ldots & 0 & -1 & 1 \end{bmatrix}.$$

On the other hand, the canonical toric ambient variety $\widetilde{Z}_{t+1}$ of $\widetilde{X}_{t+1}$ is obtained from $\widetilde{Z}_t$ by contracting the divisor corresponding to the ray ϱ_-. Hence P' is as well the primitive generator matrix for the fan of $\widetilde{Z}_{t+1}$. We conclude

$$\mathrm{Cl}(\widetilde{X}_{t+1}) \ = \ \mathbb{Z}^{r+1}/\ \mathrm{im}((P')^*) \ = \ \mathrm{Cl}(X') \ = \ K.$$

Similarly, we compare the Cox rings of $\widetilde{X}_{t+1}$ and X'. Let $\widetilde{Z}_t$ denote the canonical toric ambient variety of $\widetilde{X}_t$. Then the projection $\mathbb{Z}^{r+2} \to \mathbb{Z}^{r+1}$ defines a lift of $\widetilde{Z}_t \to \widetilde{Z}_{t+1}$ to the toric characteristic spaces and thus leads to the commutative diagram

$$\begin{tikzcd}
\widetilde{\pi}^\sharp(\widetilde{X}_t) \arrow[d, "\widetilde{\pi}"'] & \subseteq & \widetilde{W}_t \arrow[r] \arrow[d, "\widetilde{\pi}"'] & \widetilde{W}_{t+1} \arrow[d, "\pi"] & \supseteq & \pi^\sharp(\widetilde{X}_{t+1}) \arrow[d, "\pi"] \\
\widetilde{X}_t & \subseteq & \widetilde{Z}_t \arrow[r] & \widetilde{Z}_{t+1} & \supseteq & \widetilde{X}_{t+1}
\end{tikzcd}$$

where the proper transforms $\widetilde{\pi}^\sharp(\widetilde{X}_t)$ and $\pi^\sharp(\widetilde{X}_{t+1})$ are the characteristic spaces of $\widetilde{X}_t$ and $\widetilde{X}_{t+1}$ respectively and the first is mapped onto the second one. We conclude that the Cox ring of $\widetilde{X}_{t+1}$ is $R[S^+]$ graded by $\deg(T_i) = w_i$ for $i = 1, \ldots, r$ and $\deg(S^+) = w_r$ and thus is isomorphic to the Cox ring R' of X'.

The final step is to compare the defining bunches of cones $\widetilde{\Phi}_{t+1}$ of $\widetilde{X}_{t+1}$ and Φ' of X'. For this, observe that the fan of the toric ambient variety $\widetilde{Z}_{t+1}$ contains the cones $\widetilde{\sigma} + \varrho_+$, where $\sigma \in \Sigma$. Thus, every $\tau \in \Phi'$ belongs to $\widetilde{\Phi}_{t+1}$. We conclude

$$\mathrm{SAmple}(\widetilde{X}_{t+1}) \ \subseteq \ \mathrm{SAmple}(X').$$

Since $\widetilde{X}_{t+1}$ is $\mathbb{Q}$-factorial, its semiample cone is of full dimension. Both cones belong to the GIT-fan, hence we see that the above inclusion is in fact an equality. Thus $\widetilde{\Phi}_{t+1}$ equals Φ'. □

We return to the Fano varieties of Theorem 2.1.2. We first list the (finitely many) examples which do not allow duplication of a free weight and then present the starting models for constructing the Fano varieties via duplication of weights.

Proposition 2.2.4. *The varieties of Theorem 2.1.2 containing no divisors with infinite general isotropy are precisely the following ones.*

No.	$\mathcal{R}(X)$	$[w_1, \ldots, w_r]$	$-\mathcal{K}_X$	$\dim(X)$
1	$\frac{\mathbb{K}[T_1,\ldots,T_7]}{\langle T_1T_2T_3^2+T_4T_5+T_6T_7\rangle}$	$\begin{bmatrix} 0&0&1&1&1&1&1 \\ 1&1&0&1&1&1&1 \end{bmatrix}$	$\begin{bmatrix} 3 \\ 4 \end{bmatrix}$	4
2	$\frac{\mathbb{K}[T_1,\ldots,T_7]}{\langle T_1T_2T_3+T_4T_5+T_6T_7\rangle}$	$\begin{bmatrix} 0&0&1&1&0&1&0 \\ 1&1&0&1&1&1&1 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 4 \end{bmatrix}$	4
3	$\frac{\mathbb{K}[T_1,\ldots,T_6]}{\langle T_1T_2T_3^2+T_4T_5+T_6^2\rangle}$	$\begin{bmatrix} 0&0&1&1&1&1 \\ 1&1&0&1&1&1 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$	3
4.A	$\frac{\mathbb{K}[T_1,\ldots,T_6]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\begin{bmatrix} 0&1&0&1&0&1 \\ 1&0&1&0&1&0 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 2 \end{bmatrix}$	3
4.B	$\frac{\mathbb{K}[T_1,\ldots,T_6]}{\langle T_1T_2^2+T_3T_4+T_5T_6\rangle}$	$\begin{bmatrix} 0&1&1&1&1&1 \\ 1&0&1&0&1&0 \end{bmatrix}$	$\begin{bmatrix} 3 \\ 2 \end{bmatrix}$	3
4.C	$\frac{\mathbb{K}[T_1,\ldots,T_6]}{\langle T_1T_2^2+T_3T_4^2+T_5T_6^2\rangle}$	$\begin{bmatrix} 0&1&0&1&0&1 \\ 1&0&1&0&1&0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$	3

13	$\dfrac{\mathbb{K}[T_1,\dots,T_8]}{\left\langle \begin{array}{c} T_1T_2+T_3T_4+T_5T_6, \\ \lambda T_3T_4+T_5T_6+T_7T_8 \end{array} \right\rangle}$ $\lambda\in\mathbb{K}^*\setminus\{1\}$	$\begin{bmatrix} 1&0&1&0&1&0&1&0 \\ 0&1&0&1&0&1&0&1 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 2 \end{bmatrix}$	4

Proof. For a T-variety $X = X(A,P,u)$, the divisors having infinite general T-isotropy are precisely the vanishing sets of the variable S_k. Thus we just have to pick out the cases with $m=0$ from Theorem 2.1.2. □

Theorem 2.2.5. *Let X be a smooth rational Fano variety with a torus action of complexity one and Picard number two. If there is a prime divisor with infinite general isotropy on X, then X arises via iterated duplication of the free weight w_r from one of the following varieties Y.*

No.	$\mathcal{R}(Y)$	$[w_1,\dots,w_r]$	u	$\dim(Y)$
4.A	$\dfrac{\mathbb{K}[T_1,\dots,T_6,S_1]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cccccc\|c} 0&1&0&1&0&1&0 \\ 1&0&1&0&1&0&1 \end{array}\right]$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	4
4.A	$\dfrac{\mathbb{K}[T_1,\dots,T_6,S_1,S_2]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cccccc\|cc} 0&1&0&1&0&1&-1&0 \\ 1&0&1&0&1&0&1&1 \end{array}\right]$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	5
4.B	$\dfrac{\mathbb{K}[T_1,\dots,T_6,S_1]}{\langle T_1T_2^2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cccccc\|c} 0&1&1&1&1&1&1 \\ 1&0&1&0&1&0&1 \end{array}\right]$	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$	4
4.C	$\dfrac{\mathbb{K}[T_1,\dots,T_6,S_1]}{\langle T_1T_2^2+T_3T_4^2+T_5T_6^2\rangle}$	$\left[\begin{array}{cccccc\|c} 0&1&0&1&0&1&0 \\ 1&0&1&0&1&0&1 \end{array}\right]$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	4
5	$\dfrac{\mathbb{K}[T_1,\dots,T_6,S_1]}{\langle T_1T_2+T_3^2T_4+T_5^2T_6\rangle}$	$\left[\begin{array}{cccccc\|c} 0&2a+1&a&1&a&1&1 \\ 1&1&1&0&1&0&0 \end{array}\right]$ $a\geq 0$	$\begin{bmatrix} 2a+2 \\ 1 \end{bmatrix}$	4
6	$\dfrac{\mathbb{K}[T_1,\dots,T_6,S_1]}{\langle T_1T_2+T_3T_4+T_5^2T_6\rangle}$	$\left[\begin{array}{cccccc\|c} 0&2c+1&a&b&c&1&1 \\ 1&1&1&1&1&0&0 \end{array}\right]$ $a,b,c\geq 0,\quad a<b,$ $a+b=2c+1$	$\begin{bmatrix} 2c+2 \\ 1 \end{bmatrix}$	4
7	$\dfrac{\mathbb{K}[T_1,\dots,T_6,S_1]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cccccc\|c} 0&0&0&0&-1&1&1 \\ 1&1&1&1&1&1&0 \end{array}\right]$	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$	4
8	$\dfrac{\mathbb{K}[T_1,\dots,T_6,S_1,S_2]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cccccc\|cc} 0&0&0&0&0&0&1&1 \\ 1&1&1&1&1&1&0&a \end{array}\right]$ $a\in\{1,2,3\}$	$\begin{bmatrix} 1 \\ a+1 \end{bmatrix}$	5
8	$\dfrac{\mathbb{K}[T_1,\dots,T_6,S_1,S_2,S_3]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cccccc\|ccc} 0&0&0&0&0&0&1&1&1 \\ 1&1&1&1&1&1&0&a-1&a \end{array}\right]$ $a\in\{1,2\}$	$\begin{bmatrix} 1 \\ a+1 \end{bmatrix}$	6
8	$\dfrac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_4]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cccccc\|cccc} 0&0&0&0&0&0&1&1&1&1 \\ 1&1&1&1&1&1&0&0&0&1 \end{array}\right]$	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$	7
9	$\dfrac{\mathbb{K}[T_1,\dots,T_6,S_1,S_2]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cccc\|cc} 0&a_2&\dots&a_6&1&1 \\ 1&1&\dots&1&0&0 \end{array}\right]$ $0\leq a_3\leq a_5\leq a_6\leq a_4\leq a_2,$ $a_2=a_3+a_4=a_5+a_6$	$\begin{bmatrix} a_2+1 \\ 1 \end{bmatrix}$	5
10	$\dfrac{\mathbb{K}[T_1,\dots,T_5,S_1]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{ccccc\|c} 1&1&1&1&1&0 \\ -1&1&0&0&0&1 \end{array}\right]$	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$	3
11	$\dfrac{\mathbb{K}[T_1,\dots,T_5,S_1,S_2]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{ccccc\|cc} 1&1&1&1&1&0&a \\ 0&0&0&0&0&1&1 \end{array}\right]$ $a\in\{1,2\}$	$\begin{bmatrix} a+1 \\ 1 \end{bmatrix}$	4
11	$\dfrac{\mathbb{K}[T_1,\dots,T_5,S_1,S_2,S_3]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{ccccc\|ccc} 1&1&1&1&1&0&0&1 \\ 0&0&0&0&0&1&1&1 \end{array}\right]$	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$	5
12	$\dfrac{\mathbb{K}[T_1,\dots,T_5,S_1,S_2]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{ccccc\|cc} 1&1&1&1&1&0&0 \\ 0&2c&a&b&c&1&1 \end{array}\right]$ $0\leq a\leq c\leq b,\ a+b=2c$	$\begin{bmatrix} 1 \\ 2c+1 \end{bmatrix}$	4

For Nos. 4, 8 and 11, the variety Y is Fano and any iterated duplication of w_r produces a Fano variety X. For the remaining cases, the following table tells which Y are Fano and gives the characterizing condition when an iterated duplication of w_r produces a Fano variety X:

No.	*5*	*6*	*7*	*9*	*10*	*12*
Y *Fano*	$a = 0$	$c = 0$	✓	$a_2 = 0$	✓	$c = 0$
X *Fano*	$m > 2a$	$m > 3c + 1$	$m \leq 3$	$m > 2a_2$	$m \leq 2$	$m > 3c$

Proof. A T-variety $X = X(A, P, u)$ has a divisor with infinite general T-isotropy if and only if $m \geq 1$ holds. In the cases 4.A, 4.B, 4.C, 5, 6, 7, 9, 10 and 12 we directly infer from Theorem 2.1.2 that the examples with higher m arise from those listed in the table above via iterated duplication of w_r.

We still have to consider Nos. 8 and 11. If X is a variety of type 8, then the condition for X to be a Fano variety is

$$4 + a_2 + \ldots, +a_m > ma_m,$$

where $a_m = 1, 2, 3$ and $0 \leq a_2 \leq \ldots \leq a_m$. This is satisfied if and only if one of the following conditions holds:

(i) $a_2 = \ldots = a_m \in \{1, 2, 3\}$.
(ii) $a_2 + 1 = a_3 \ldots = a_m \in \{1, 2\}$, with $m \geq 3$.
(iii) $a_2 = a_3 = 0$ and $a_4 = \ldots = a_m = 1$, with $m \geq 4$.

Similarly for No. 11 the Fano condition in the table of Theorem 2.1.2 is equivalent to the fulfillment of one of the following:

(i) $a_2 = \ldots = a_m \in \{1, 2\}$.
(ii) $a_2 = 0$ and $a_3 = \ldots = a_m = 1$, with $m \geq 3$.

In both cases this explicit characterization makes clear that we are in the setting of the duplication of a free weight. □

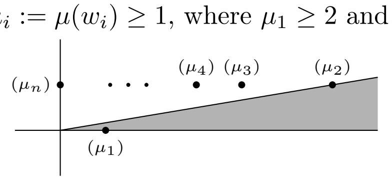
Remark 2.2.6. Consider iterated duplication of w_r for a variety $X = X(A, P, u)$ as in Theorem 2.2.5. Recall that the effective cone of X is decomposed as $\tau^+ \cup \tau_X \cup \tau^-$, where $\tau_X = \mathrm{Ample}(X)$. Lemma 2.4.11 (i) says $w_r \notin \tau_X$ and thus we have a unique $\kappa \in \{\tau^+, \tau^-\}$ with $w_r \notin \kappa$. Then the number of flips per duplication step equals

$$|\{\mathrm{cone}(w_{ij}), \mathrm{cone}(w_k);\ w_{ij}, w_k \in \kappa\}| - 1.$$

In particular, for Nos. 4.A, 4.B, 4.C, 8, 11, 9 with $a_i = 0$, 12 with $b = 0$ the duplication steps require no flips.

Remark 2.2.7. For toric Fano varieties, there is no statement like Corollary 2.1.3. Recall from [**10**] that all smooth projective toric varieties Z with $\mathrm{Cl}(Z) = \mathbb{Z}^2$ admit a description via the following data:

- weights $w_1 := (1, 0)$ and $w_i := (b_i, 1)$ with $0 = b_n < b_{n-1} < \ldots < b_2$,
- multiplicities $\mu_i := \mu(w_i) \geq 1$, where $\mu_1 \geq 2$ and $\mu_2 + \ldots + \mu_n \geq 2$ hold.



The variety Z arises from the bunched polynomial ring $(R, \mathfrak{F}, \Phi)$, where R equals $\mathbb{K}[S_{ij};\ 1 \leq i \leq n, 1 \leq j \leq \mu_i]$ with the system of generators $\mathfrak{F} = (S_{11}, \ldots, S_{n\mu_n})$, generator degrees $\deg(S_{ij}) = w_i$ and the bunch $\Phi = \{\mathrm{cone}(w_1, w_i);\ i = 2, \ldots, n\}$. In this setting Z is Fano if and only if

$$b_2(\mu_2 + \ldots + \mu_n) < \mu_1 + \mu_2 b_2 + \ldots + \mu_{n-1} b_{n-1}.$$

For any $n \in \mathbb{Z}_{\geq 4}$ and $i = 2, \ldots, n$ set $\mu_i := 1$ and $w_i := (n - i, 1)$. Then, with $\mu_1 := 2$ we obtain a smooth (non-Fano) toric variety Z'_n of Picard number two and dimension $n - 1$. Moreover, for $\mu_1 := 1 + (n - 2)(n - 1)/2$ we obtain a smooth toric Fano variety Z_n of Picard number two that is Fano and is obtained from Z'_n via iterated duplication of w_1 but cannot be constructed from any lower dimensional smooth variety this way.

2.3. Geometry of the Fano varieties

We take a closer look at the Fano varieties listed in Theorem 2.1.2 and prove that they fulfill Mukai's conjecture, see Proposition 2.3.6. Moreover, we describe explicitly their Mori fiber spaces and their divisorial contractions. The approach uses suitable toric ambient varieties. The following Remark can be found, at least partially, for example in [**21**, Section 7.3].

Remark 2.3.1. Let Z be a smooth projective toric variety of Picard number two, given by weight vectors $w_1 := (1,0)$ and $w_i := (b_i, 1)$ with $0 = b_n < b_{n-1} < \ldots < b_2$, and multiplicities $\mu_i := \mu(w_i) \geq 1$, where $\mu_1 \geq 2$ and $\mu_2 + \ldots + \mu_n \geq 2$ as in Remark 2.2.7. Then the toric variety Z is a projectivized split vector bundle of rank r over a projective space $\mathbb{P}_s$, where $s := \mu_1 - 1$ and $r := \mu_2 + \ldots + \mu_n - 1$. More precisely, we have

$$Z \;\cong\; \mathbb{P}\left(\bigoplus_{i=1}^{\mu_n} \mathcal{O}_{\mathbb{P}_s} \oplus \bigoplus_{i=1}^{\mu_{n-1}} \mathcal{O}_{\mathbb{P}_s}(b_{n-1}) \oplus \ldots \oplus \bigoplus_{i=1}^{\mu_2} \mathcal{O}_{\mathbb{P}_s}(b_2)\right).$$

The bundle projection $Z \to \mathbb{P}_s$ is the elementary contraction associated to the divisor class $w_1 \in \mathbb{Z}^2 = \mathrm{Cl}(Z)$. If $n = 2$ holds, then we have $Z \cong \mathbb{P}_s \times \mathbb{P}_r$. If $n = 3$ and $\mu_3 = 1$ hold, then the class $w_3 \in \mathbb{Z}^2 = \mathrm{Cl}(Z)$ gives rise to a divisorial contraction onto a weighted projective space:

$$Z \;\to\; Z' \;:=\; \mathbb{P}(\underbrace{1, \ldots, 1}_{\mu_1}, \underbrace{b_2, \ldots, b_2}_{\mu_2}).$$

The exceptional divisor $E_Z \subseteq Z$ is isomorphic to $\mathbb{P}_s \times \mathbb{P}_{\mu_2 - 1}$ and the center $C(Z') \subseteq Z'$ of the contraction is isomorphic to $\mathbb{P}_{\mu_2 - 1}$. In particular, for $\mu_2 = 1$, we have $E_Z \cong \mathbb{P}_s$ and $C(Z')$ is a point.

From the explicit description of the Cox ring of our Fano variety X, we obtain via Construction 1.4.4 a closed embedding $X \to Z$ into a toric variety Z. As a byproduct of our classification, it turns out that, whenever X admits a elementary contraction, then X inherits all its elementary contractions from Z. Remark 2.3.1 together with the explicit equations for X in Z will then allow us to study the situation in detail. We now present the results. The cases are numbered according to the table of Theorem 2.1.2. Moreover, we denote by $Q_3 \subseteq \mathbb{P}_4$ and $Q_4 \subseteq \mathbb{P}_5$ the three and four-dimensional smooth projective quadrics and we write $\mathbb{P}(a_1^{\mu_1}, \ldots, a_r^{\mu_r})$ for the weighted projective space, where the superscript μ_i indicates that the weight a_i occurs μ_i times.

No. 1 The variety X is of dimension four and admits two elementary contractions, $Q_4 \leftarrow X \to \mathbb{P}_1$. The morphism $X \to Q_4$ is a divisorial contraction with exceptional divisor isomorphic to $\mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1$ and center isomorphic to $\mathbb{P}_1 \times \mathbb{P}_1$. The morphism $X \to \mathbb{P}_1$ is a Mori fiber space with general fiber isomorphic to Q_3 and singular fibers over $[0,1]$ and $[1,0]$ each isomorphic to the singular quadric $V(T_2T_3 + T_4T_5) \subseteq \mathbb{P}_4$.

No. 2 The variety X is of dimension four and admits two elementary contractions, $Q_4 \leftarrow X \to \mathbb{P}_3$. The morphism $X \to Q_4$ is a divisorial contraction with exceptional divisor isomorphic to a hypersurface of bidegree $(1,1)$ in $\mathbb{P}_1 \times \mathbb{P}_3$ and center isomorphic to $\mathbb{P}_1$. The morphism $X \to \mathbb{P}_3$ is a Mori fiber space with fibers isomorphic to $\mathbb{P}_1$.

No. 3 The variety X is of dimension three and occurs as No. 2.29 in the Mori-Mukai classification [**53**]. Moreover, X admits two elementary contractions, $Q_3 \leftarrow X \to \mathbb{P}_1$. The morphism $X \to Q_3$ is a divisorial contraction with exceptional divisor isomorphic to $\mathbb{P}_1 \times \mathbb{P}_1$ and center isomorphic to $\mathbb{P}_1$. The morphism $X \to \mathbb{P}_1$ is a Mori fiber space with general fiber isomorphic to $\mathbb{P}_1 \times \mathbb{P}_1$ and singular fibers over $[0,1]$ and $[1,0]$ each isomorphic to $V(T_1T_2 + T_3^2) \subseteq \mathbb{P}_3$.

No. 4A *Case 1:* we have $c = -1$. Then X admits two elementary contractions $Y \leftarrow X \rightarrow \mathbb{P}_2$, where $Y := V(T_1T_2 + T_3T_4 + T_5T_6) \subseteq \mathbb{P}_{m+4}$ is a terminal factorial Fano variety which is smooth if and only if $m = 1$ holds. The morphism $X \rightarrow Y$ is a divisorial contraction with exceptional divisor isomorphic to a hypersurface of bidegree $(1,1)$ in $\mathbb{P}_2 \times \mathbb{P}_{m+1}$ and center isomorphic to $\mathbb{P}_{m+1}$. The morphism $X \rightarrow \mathbb{P}_2$ is a Mori fiber space with fibers isomorphic to $\mathbb{P}_{m+1}$.

Case 2: we have $c = 0$. Then X is a hypersurface of bidegree $(1,1)$ in $\mathbb{P}_2 \times \mathbb{P}_{m+2}$. Moreover, X admits two Mori fiber spaces $\mathbb{P}_{m+2} \leftarrow X \rightarrow \mathbb{P}_2$. The Mori fiber space $X \rightarrow \mathbb{P}_2$ has fibers isomorphic to $\mathbb{P}_{m+1}$, whereas the Mori fiber space $X \rightarrow \mathbb{P}_{m+1}$ has general fiber isomorphic to $\mathbb{P}_1$ and special fibers over $V(T_1, T_2, T_3) \subseteq \mathbb{P}_{m+2}$ isomorphic to $\mathbb{P}_2$. For $m = 0$, we have $\dim(X) = 3$ and X is the variety No. 2.32 in [**53**].

No. 4B The variety X admits two elementary contractions $Y \leftarrow X \rightarrow \mathbb{P}_2$, where $Y := V(T_1^2 + T_2T_3 + T_4T_5) \subseteq \mathbb{P}_{m+4}$ is a terminal factorial Fano variety. The variety Y is smooth if and only if $m = 0$ holds and in this case X occurs as No. 2.31 in [**53**]. The morphism $X \rightarrow Y$ is a divisorial contraction with exceptional divisor isomorphic to a hypersurface of bidegree $(1,1)$ in $\mathbb{P}_2 \times \mathbb{P}_{m+1}$ and center isomorphic to $\mathbb{P}_{m+1}$. The morphism $X \rightarrow \mathbb{P}_2$ is a Mori fiber space with fibers isomorphic to $\mathbb{P}_{m+1}$.

No. 4C The variety X is a hypersurface of bidegree $(2,1)$ in $\mathbb{P}_2 \times \mathbb{P}_{m+2}$; for $m = 0$ we have $\dim(X) = 3$ and X is No. 2.24 in [**53**]. Moreover, X admits two Mori fiber spaces $\mathbb{P}_{m+2} \leftarrow X \rightarrow \mathbb{P}_2$. The morphism $X \rightarrow \mathbb{P}_2$ has fibers isomorphic to $\mathbb{P}_{m+1}$. To describe the fibers of $\varphi\colon X \rightarrow \mathbb{P}_{m+2}$, set $Y_i := V_{\mathbb{P}_{m+2}}(T_i)$, $Y_{ij} := V_{\mathbb{P}_{m+2}}(T_i, T_j)$ and $Y_{123} := V_{\mathbb{P}_{m+2}}(T_1, T_2, T_3)$. Then we have

$$\varphi^{-1}(z) \ \cong \ \begin{cases} \mathbb{P}_2 & \text{if } z \in Y_{123}, \\ \mathbb{P}_1 & \text{if } z \in (Y_{12} \cup Y_{13} \cup Y_{23}) \setminus Y_{123}, \\ V_{\mathbb{P}_2}(T_1T_2) & \text{if } z \in (Y_1 \cup Y_2 \cup Y_3) \setminus (Y_{12} \cup Y_{13} \cup Y_{23}), \\ \mathbb{P}_1 & \text{otherwise.} \end{cases}$$

No. 5 The variety X admits a Mori fiber space $\varphi\colon X \rightarrow \mathbb{P}_{m+1}$, whose general fiber is isomorphic to $\mathbb{P}_1 \times \mathbb{P}_1$. More precisely, with $Y_1 := V_{\mathbb{P}_{m+1}}(T_1)$ and $Y_2 := V_{\mathbb{P}_{m+1}}(T_2)$, we have

$$\varphi^{-1}(z) \ \cong \ \begin{cases} V_{\mathbb{P}_3}(T_1T_2) & \text{if } z \in Y_1 \cap Y_2, \\ V_{\mathbb{P}_3}(T_1T_2 + T_3^2) & \text{if } z \in Y_1 \setminus Y_2 \text{ or } z \in Y_2 \setminus Y_1, \\ \mathbb{P}_1 \times \mathbb{P}_1 & \text{otherwise.} \end{cases}$$

No. 6 The variety X admits a Mori fiber space $X \rightarrow \mathbb{P}_m$, with general fiber isomorphic to Q_3 and singular fibers over $V(T_1) \subseteq \mathbb{P}_m$ each isomorphic to the hypersurface $V(T_1T_2 + T_3T_4) \subseteq \mathbb{P}_4$.

No. 7 The variety X admits a divisorial contraction $X \rightarrow \mathbb{P}_{m+3}$ with exceptional divisor isomorphic to the projectivized split bundle

$$\mathbb{P}\left(\bigoplus_{i=1}^{m} \mathcal{O}_{\mathbb{P}_1 \times \mathbb{P}_1} \oplus \mathcal{O}_{\mathbb{P}_1 \times \mathbb{P}_1}(1,1) \right)$$

and center isomorphic to $\mathbb{P}_1 \times \mathbb{P}_1$. Moreover, if $m = 1$ holds, X admits a further divisorial contraction $X \rightarrow Q_4$ with exceptional divisor isomorphic to $\mathbb{P}_3$ and center a point.

No. 8 Here we have $X = \mathbb{P}(\mathcal{O}_{Q_4} \oplus \mathcal{O}_{Q_4}(a_2) \ldots \oplus \mathcal{O}_{Q_4}(a_m))$. Thus, there is a Mori fiber space $X \rightarrow Q_4$ with fibers isomorphic to $\mathbb{P}_{m-1}$. If $a_2 = \ldots = a_m > 0$ holds, then X admits in addition a divisorial contraction $X \rightarrow Y$, where $Y :=$

$V(T_1T_2 + T_3T_4 + T_5T_6) \subseteq \mathbb{P}(1^6, a_2^{m-1})$. The exceptional divisor is isomorphic to $Q_4 \times \mathbb{P}_{m-2}$ and the center to $\mathbb{P}_{m-2}$.

No. 9 The variety X is a bundle over $\mathbb{P}_{m-1}$ with fibers isomorphic to Q_4. In particular, if $a_i = 0$ holds for all $2 \le i \le 6$, then $X \cong Q_4 \times \mathbb{P}_{m-1}$.

No. 10 The variety X admits a divisorial contraction $X \to \mathbb{P}_{m+2}$ with exceptional divisor isomorphic to the projectivized split bundle

$$\mathbb{P}\left(\bigoplus_{i=1}^{m} \mathcal{O}_{\mathbb{P}_1} \oplus \mathcal{O}_{\mathbb{P}_1}(1)\right)$$

and center isomorphic to $\mathbb{P}_1$. For $m = 1$, we have $\dim(X) = 3$ and X is No. 2.30 from [**53**]; in this case it admits a further divisorial contraction $X \to Q_3$ with exceptional divisor isomorphic to $\mathbb{P}_2$ and center a point.

No. 11 Here $X = \mathbb{P}(\mathcal{O}_{Q_3} \oplus \mathcal{O}_{Q_3}(a_2) \ldots \oplus \mathcal{O}_{Q_3}(a_m))$ holds. Thus, there is a Mori fiber space $X \to Q_3$ with fibers isomorphic to $\mathbb{P}_{m-1}$. If $a_2 = \ldots = a_m > 0$ holds, then X admits a divisorial contraction $X \to Y$, where the variety Y equals $V(T_1T_2 + T_3T_4 + T_5^2) \subseteq \mathbb{P}(1^5, a_2^{m-1})$. The exceptional divisor is isomorphic to $Q_3 \times \mathbb{P}_{m-2}$ and the center to $\mathbb{P}_{m-2}$.

No. 12 The variety X is a bundle over $\mathbb{P}_{m-1}$ with fibers isomorphic to Q_3. In particular, if $a = b = c = 0$ holds, then $X \cong Q_3 \times \mathbb{P}_{m-1}$.

No. 13 This case presents a one-parameter family of varieties X_λ, with parameter $\lambda \in \mathbb{K}^*\backslash\{1\}$. They are generally non-isomorphic to each other, except for the pairs $X_\lambda \cong X_{\lambda^{-1}}$ for all λ. The variety X_λ is the intersection of two hypersurfaces

$$D_1 \;=\; V(T_1S_1 + T_2S_2 + T_3S_3), \qquad D_2 \;=\; V(\lambda T_2S_2 + T_3S_3 + T_4S_4),$$

both of bidegree (1,1) in $\mathbb{P}_3 \times \mathbb{P}_3$, where the T_i are the coordinates of the first $\mathbb{P}_3$ and the S_j those of the second. Note that each D_i has an isolated singularity, which is not contained in the other hypersurface. Both D_1, D_2 are terminal and factorial. Moreover, X admits two Mori fiber spaces $\mathbb{P}_3 \leftarrow X \to \mathbb{P}_3$, both with typical fiber $\mathbb{P}_1$ and having four special fibers, all isomorphic to $\mathbb{P}_2$ and lying over the points $[1,0,0,0]$, $[0,1,0,0]$, $[0,0,1,0]$ and $[0,0,0,1]$.

Remark 2.3.2. In contrast to the toric case, a smooth projective variety of Picard number 2 with torus action of complexity one need not admit a non-trivial Mori fiber space. For example, in Theorem 2.1.2, this happens in precisely two cases, namely No. 7 and No. 10, both with $m = 1$.

Remark 2.3.3. In the list of Theorem 2.1.2 there are several examples, where the effective cone coincides with the cone of movable divisor classes: No. 4A with $c = 0$, No. 4C, No. 5 with $a = 0$, No. 6 with $a = 0$, No. 8 with $a_2 = 0$, No. 9 with $a_3 = 0$, No. 11 with $a_2 = 0$, No. 12 with $a = 0$ and No. 13. Thus, these varieties admit no divisorial contraction.

Remark 2.3.4. In Theorem 2.1.1 it is possible that non-isomorphic varieties share the same Cox ring and thus differ from each other by a small quasimodification, i.e. only by the choice of the ample class. This happens exactly in the following cases:

(i) No. 4 with $l_2 = l_4 = 2$, $l_6 = 1$, $a = 0$, $b = 1$, $c_i = 0$ for all $i = 1, \ldots, m$ has the same Cox ring as No. 5 with $a = 0$. Note that for $m = 0$ both varieties are truly almost Fano, whereas for $m \ge 1$ No. 5 is Fano.

(ii) For $m \ge 1$, No. 4 with $l_2 = 2$, $l_4 = l_6 = 1$, $a = b = 1$, $c_i = 0$ for all $i = 1, \ldots, m$ has the same Cox ring as No. 6 with $a = c = 0$ and $b = 1$. Note that for $m = 1$ both varieties are truly almost Fano, whereas for $m \ge 2$ No. 6 is Fano.

(iii) For $m \geq 2$, No. 7 has the same Cox ring as No. 9 with $a_2 = 2$ and $a_3 = \ldots = a_6 = 1$. Note that for $m = 2, 3$ No. 7 is Fano, for $m = 4$ both varieties are truly almost Fano, whereas for $m \geq 5$ No. 9 is Fano.

(iv) For $m \geq 2$, No. 10 has the same Cox ring as No. 12 with $a = b = c = 1$. Note that for $m = 2$ No. 10 is Fano, for $m = 3$ both varieties are truly almost Fano, whereas for $m \geq 4$ No. 12 is Fano.

Mukai's conjecture was proven for Fano varieties of dimension at most five and for toric Fano varieties of arbitrary dimension, see [**13, 1, 20**] for details. As an application of our classification results, we prove Mukai's conjecture for smooth rational non-toric Fano complexity one varieties of Picard number at most two.

Conjecture 2.3.5. (Mukai's Conjecture [54]) *For a Fano variety X we have*

$$\rho(X)(q(X) - 1) \ \leq \ \dim(X)$$

and equality holds if and only if X is isomorphic to the $\rho(X)$-th product of the projective space $\mathbb{P}_{q(X)-1}$.

Proposition 2.3.6. *Let X be a smooth rational non-toric Fano variety with a torus action of complexity one and Picard number at most two. Then X fulfills Mukai's conjecture, Conjecture 2.3.5.*

Proof. In Picard number one, there are by a result of Liendo and Süß [**49**, Thm. 6.5] up to isomorphism just two smooth rational non-toric Fano complexity one varieties, namely the a three and a four dimensional intrinsic quadric. The case of smooth intrinsic quadrics will be settled in Proposition 3.2.14 which we will prove in Chapter three.

In Picard number two, all smooth rational non-toric Fano complexity one varieties X are listed in the table of Theorem 2.1.2. Note that No. 4.A and Nos. 7–12 are smooth intrinsic quadrics, i.e. those varieties fulfill Mukai's conjecture by Proposition 3.2.14. It remains to settle the remaining numbers.

For Nos. 1, 2, 3, 4.B, 4.C and for No. 13 one can directly read off the Picard index $q(X)$ from the table in Theorem 2.1.2 and thereby check that X fulfills Mukai's conjecture. For X arising from Nos. 5 and 6, we have $q(X) \leq 2$ and $q(X) \leq 3$ as well as $\dim(X) \geq 4$ and $\dim(X) \geq 5$, respectively. We conclude that X fulfills Mukai's conjecture. ☐

2.4. First structural constraints

As a first step towards the proof of our classification results stated in Section 2.1, we derive constraints on the defining matrices of smooth rational varieties with a torus action of complexity one having Picard number two. We work in the notation of Section 1.4. The aim is to show the following.

Proposition 2.4.1. *Let X be a non-toric smooth rational projective variety with a torus action of complexity one and Picard number $\rho(X) = 2$. Then $X \cong X(A, P, u)$, where P is irredundant and fits into one of the following cases:*

- *(I) We have $r = 2$ and one of the following constellations:*
  - *(a) $m \geq 0$ and $n = 4 + n_0$, where $n_0 \geq 3$, $n_1 = n_2 = 2$.*
  - *(b) $m = 0$ and $n = 6$, where $n_0 = 3$, $n_1 = 2$, $n_2 = 1$.*
  - *(c) $m = 0$ and $n = 5$, where $n_0 = 3$, $n_1 = 1$, $n_2 = 1$.*
  - *(d) $m \geq 0$ and $n = 6$, where $n_0 = n_1 = n_2 = 2$.*
  - *(e) $m \geq 0$ and $n = 5$, where $n_0 = n_1 = 2$, $n_2 = 1$.*
  - *(f) $m \geq 1$ and $n = 4$, where $n_0 = 2$, $n_1 = n_2 = 1$.*
- *(II) We have $r = 3$ and one of the following constellations:*
  - *(a) $m = 0$ and $n = 8$, where $n_0 = n_1 = n_2 = n_3 = 2$.*

(b) $m = 0$ *and* $n = 7$*, where* $n_0 = n_1 = n_2 = 2$*,* $n_3 = 1$*.*
(c) $m = 0$ *and* $n = 6$*, where* $n_0 = n_1 = 2$*,* $n_2 = n_3 = 1$*.*

The statement is an immediate consequence of Propositions 2.4.12 and 2.4.13; see end of this section. Throughout the whole section, the defining matrix P is irredundant. In particular, $X(A, P, u)$ is non-toric if and only if $r \geq 2$ holds, i.e. we have a relation in the Cox ring.

We first study the impact of $X = X(A, P, u)$ being locally factorial on the defining matrix P, where locally factorial means that the local rings of the points $x \in X$ are unique factorization domains.

Lemma 2.4.2. *Let* $X = X(A, P, u)$ *be non-toric and locally factorial. If* X *is weakly tropical, then* $n_i \geq 2$ *holds for all* $i = 0, \ldots, r$*.*

Proof. Assume that $n_i = 1$ holds for some i. Since X is weakly tropical, there exists a cone $\sigma \in \Sigma_X$ of dimension $s + 1$ contained in the leaf λ_i. Because of $n_i = 1$ we have $\sigma = \varrho_{i1} + \tau$ with a face $\tau \preceq \sigma$ such that $\tau \subseteq \lambda$. Now, $\sigma = P(\gamma_0^*)$ holds for some $\gamma_0 \subseteq \mathrm{rlv}(u)$. Since the points of $X(\gamma_0)$ are factorial, σ is a regular cone. Thus, also $\tau \subseteq \lambda$ must be regular. This implies $l_{i1} = 1$, contradicting irredundancy of P. □

Lemma 2.4.3. *Let* $X = X(A, P, u)$ *be non-toric and locally factorial. If* X *is weakly tropical, then* $\rho(X) \geq r + 3$ *holds.*

Proof. Lemma 2.4.2 ensures $n_i \geq 2$ for all $i = 1, \ldots, r$, hence $n \geq 2 \cdot (r + 1)$. The s-dimensional lineality space $\lambda = \{0\} \times \mathbb{Q}^s \subseteq \mathrm{trop}(X)$ is a union of cones of Σ_X. Thus P must have at least $s + 1$ columns v_k which means $m \geq s + 1$. Together this yields

$$\rho(X) \;=\; n + m - (r - 1) - (s + 1) \;\geq\; r + 3.$$

□

Lemma 2.4.4. *Let* $X = X(A, P, u)$ *be non-toric and not weakly tropical. If* X *is* $\mathbb{Q}$*-factorial, then there is an elementary big cone in* Σ_X*.*

Proof. Since X is not weakly tropical, there exists a big cone $\sigma \in \Sigma_X$. We have $\sigma = P(\gamma_0^*)$ with $\gamma_0 \in \mathrm{rlv}(u)$. Since the points of $X(\gamma_0)$ are $\mathbb{Q}$-factorial, the cone σ is simplicial. For every $i = 0 \ldots, r$ choose a ray $\varrho_i \preceq \sigma$ with $\varrho_i \in \lambda_i$. Then $\sigma_0 := \varrho_0 + \ldots + \varrho_r \preceq \sigma$ is as wanted. □

Corollary 2.4.5. *Let* $X = X(A, P, u)$ *be non-toric and locally factorial. If* $\rho(X) \leq 4$ *holds, then there exists an elementary big cone* $\sigma \in \Sigma_X$*.*

Next we investigate the effect of quasismoothness on the defining matrix P, where we call $X = X(A, P, u)$ *quasismooth* if $\widehat{X}$ is smooth. Thus, quasismoothness means that X has at most quotient singularities by quasitori. The smoothness of $\widehat{X}$ will lead to conditions on P via the Jacobian of the defining relations of $\overline{X}$.

Remark 2.4.6. Let (A, P) be defining matrices. Then the Jacobian J_g of the defining relations $g_0, \ldots, g_{r-2}$ is of the shape $J_g = (J, 0)$ with a zero block of size $(r - 1) \times m$ corresponding to the variables $S_1, \ldots, S_m$ and a block

$$J \;:=\; \left[\begin{array}{ccccccccc} \delta_{10} & \delta_{11} & \delta_{12} & 0 & & & & & \\ 0 & \delta_{21} & \delta_{22} & \delta_{23} & 0 & & & & \\ & & & & \vdots & & & & \\ & & & & & \delta_{r-2,r-3} & \delta_{r-2,r-2} & \delta_{r-2,r-1} & 0 \\ & & & & & 0 & \delta_{r-1,r-2} & \delta_{r-1,r-1} & \delta_{r-1,r} \end{array}\right],$$

of size $(r-1) \times n$, where each vector $\delta_{a,i}$ is a nonzero multiple of the gradient of the monomial $T_i^{l_i}$:

$$\delta_{a,i} \;=\; \alpha_{a,i}\left(l_{i1}\frac{T_i^{l_i}}{T_{i1}},\;\ldots,\; l_{in_i}\frac{T_i^{l_i}}{T_{in_i}}\right), \qquad \alpha_{a,i} \;\in\; \mathbb{K}^*.$$

For given $1 \le a, b \le r-1$, $0 \le i \le r$ and $z \in \overline{X}$, we have $\delta_{a,i}(z) = 0$ if and only if $\delta_{b,i}(z) = 0$. Moreover, the Jacobian $J_g(z)$ of a point $z \in \overline{X}$ is of full rank if and only if $\delta_{a,i}(z) = 0$ holds for at most two different $i = 0, \ldots, r$.

Lemma 2.4.7. *Assume that $X = X(A,P,u)$ is non-toric and that there is an elementary big cone $\sigma = \varrho_{0j_0} + \ldots + \varrho_{rj_r} \in \Sigma_X$. If X is quasismooth, then $l_{ij_i} \ge 2$ holds for at most two $i = 0, \ldots, r$.*

Proof. We have $\sigma = P(\gamma_0^*)$ with a relevant face $\gamma_0 \in \mathrm{rlv}(u)$. Since X is quasismooth, any $z \in \overline{X}(\gamma_0)$ is a smooth point of $\overline{X}$. Thus, $J_g(z)$ is of full rank $r-1$. Consequently, $\delta_{a,i}(z) = 0$ holds for at most two different i. This means $l_{ij_i} \ge 2$ for at most two different i. □

Corollary 2.4.8. *Let $X = X(A,P,u)$ be non-toric and quasismooth. If there is an elementary big cone in Σ_X, then $n_i = 1$ holds for at most two different $i = 0, \ldots, r$.*

Lemma 2.4.9. *Let (A,P) be defining matrices. Consider the rays $\gamma_k := \mathrm{cone}(e_k)$ and $\gamma_{ij} := \mathrm{cone}(e_{ij})$ of the orthant $\gamma \subseteq \mathbb{Q}^{r+s}$ and the two-dimensional faces*

$$\gamma_{k_1,k_2} := \gamma_{k_1} + \gamma_{k_2}, \quad \gamma_{ij,k} := \gamma_{ij} + \gamma_k, \quad \gamma_{i_1j_1,i_2j_2} := \gamma_{i_1j_1} + \gamma_{i_2j_2}.$$

(i) *All γ_k, resp. γ_{k_1,k_2}, are $\mathfrak{F}$-faces and each $\overline{X}(\gamma_k)$, resp. $\overline{X}(\gamma_{k_1,k_2})$, consists of singular points of $\overline{X}$.*

(ii) *A given γ_{ij}, resp. $\gamma_{ij,k}$, is an $\mathfrak{F}$-face if and only if $n_i \ge 2$ holds. In that case, $\overline{X}(\gamma_{ij})$, resp. $\overline{X}(\gamma_{ij,k})$, consists of smooth points of $\overline{X}$ if and only if $r = 2$, $n_i = 2$ and $l_{i,3-j} = 1$ hold.*

(iii) *A given γ_{ij_1,ij_2} with $j_1 \ne j_2$ is an $\mathfrak{F}$-face if and only if $n_i \ge 3$ holds. In that case, $\overline{X}(\gamma_{ij_1,ij_2})$ consists of smooth points of $\overline{X}$ if and only if $r = 2$, $n_i = 3$ and $l_{ij} = 1$ for the $j \ne j_1, j_2$ hold.*

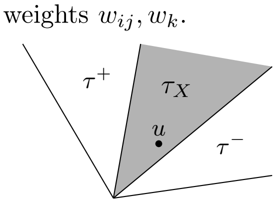
(iv) *A given $\gamma_{i_1j_1,i_2j_2}$ with $i_1 \ne i_2$ is an $\mathfrak{F}$-face if and only if we have $n_{i_1}, n_{i_2} \ge 2$ or $n_{i_1} = n_{i_2} = 1$ and $r = 2$. In the former case, $\overline{X}(\gamma_{i_1j_1,i_2j_2})$ consists of smooth points of $\overline{X}$ if and only if one of the following holds:*
   - *$r = 2$, $n_{i_t} = 2$ and $l_{i_t,3-j_t} = 1$ for a $t \in \{1,2\}$,*
   - *$r = 3$, $n_{i_1} = n_{i_2} = 2$, $l_{i_1,3-j_1} = l_{i_2,3-j_2} = 1$.*

Proof. The statements follow directly from the structure of the defining relations $g_0, \ldots, g_{r-2}$ of $R(A,P)$ and the shape of the Jacobian J_g. □

We now restrict to the case that the rational divisor class group $\mathrm{Cl}(X)_{\mathbb{Q}} = K_{\mathbb{Q}}$ of $X = X(A,P,u)$ is of dimension two. Set $\tau_X := \mathrm{Ample}(X)$. Then the effective cone $\mathrm{Eff}(X)$ is of dimension two and is uniquely decomposed into three convex sets

$$\mathrm{Eff}(X) \;=\; \tau^+ \cup \tau_X \cup \tau^-,$$

such that τ^+, τ^- do not intersect the ample cone τ_X and $\tau^+ \cap \tau^-$ consists of the origin. Recall that $u \in \tau_X$ holds and that, due to $\tau_X \subseteq \mathrm{Mov}(X)$, each of τ^+ and τ^- contains at least two of the weights w_{ij}, w_k.



Remark 2.4.10. Consider $X = X(A, P, u)$ such that $\mathrm{Cl}(X)_{\mathbb{Q}}$ is of dimension two. Then, for every $\mathfrak{F}$-face $\{0\} \neq \gamma_0 \preceq \gamma$ precisely one of the following inclusions holds

$$Q(\gamma_0) \ \subseteq \ \tau^+, \qquad \tau_X \ \subseteq \ Q(\gamma_0)^\circ, \qquad Q(\gamma_0) \ \subseteq \ \tau^-.$$

The $\mathfrak{F}$-faces $\gamma_0 \preceq \gamma$ satisfying the second inclusion are exactly those with $\gamma_0 \in \mathrm{rlv}(u)$, i.e. the relevant ones.

In the following, we will frequently work with the canonical base vectors $e_{ij}, e_k \in E$ and the faces

$$\gamma_{i_1 j_1, \ldots i_a j_a, k_1, \ldots, k_b} \ := \ \mathrm{cone}(e_{i_1 j_1} \ldots e_{i_a j_a}, e_{k_1}, \ldots, e_{k_b}) \ \preceq \ \gamma$$

of the orthant $\gamma = \mathbb{Q}^{n+m}_{\geq 0}$.

Lemma 2.4.11. *Let $X = X(A, P, u)$ be non-toric with* $\mathrm{rk}\,(\mathrm{Cl}(X)) = 2$*.*

(i) *Suppose that X is $\mathbb{Q}$-factorial. Then $w_k \notin \tau_X$ holds for all $1 \leq k \leq m$ and for all $0 \leq i \leq r$ with $n_i \geq 2$ we have $w_{ij} \notin \tau_X$, where $1 \leq j \leq n_i$.*

(ii) *Suppose that X is quasismooth, $m > 0$ holds and there is $0 \leq i_1 \leq r$ with $n_{i_1} \geq 3$. Then the w_{ij}, w_k with $n_i \geq 3$, $j = 1, \ldots, n_i$ and $k = 1, \ldots, m$ lie either all in τ^+ or all in τ^-.*

(iii) *Suppose that X is quasismooth and there is $0 \leq i_1 \leq r$ with $n_{i_1} \geq 4$. Then the w_{ij} with $n_i \geq 4$ and $j = 1, \ldots, n_i$ lie either all in τ^+ or all in τ^-.*

(iv) *Suppose that X is quasismooth and there exist $0 \leq i_1 < i_2 \leq r$ with $n_{i_1}, n_{i_2} \geq 3$. Then the w_{ij} with $n_i \geq 3$, $j = 1, \ldots, n_i$ lie either all in τ^+ or all in τ^-.*

(v) *Suppose that X is quasismooth. Then $w_1, \ldots, w_m$ lie either all in τ^+ or all in τ^-.*

Proof. We prove (i). By Lemma 2.4.9 (i) and (ii), the rays $\gamma_k, \gamma_{ij} \preceq \gamma$ with $n_i \geq 2$ are $\mathfrak{F}$-faces. Since X is $\mathbb{Q}$-factorial, the ample cone $\tau_X \subseteq K_{\mathbb{Q}}$ of X is of dimension two and thus $\tau_X \subseteq Q(\gamma_{ij})^\circ$ or $\tau_X \subseteq Q(\gamma_k)^\circ$ is not possible. Remark 2.4.10 yields the assertion.

We turn to (ii). By Lemma 2.4.9 (i) and (ii), all $\gamma_k, \gamma_{ij}, \gamma_{ij,k} \preceq \gamma$ in question are $\mathfrak{F}$-faces and the corresponding pieces in $\overline{X}$ consist of singular points. Because X is quasismooth, none of these $\mathfrak{F}$-faces is relevant. Thus, Remark 2.4.10 gives $w_{i_1 1} \in \tau^+$ or $w_{i_1 1} \in \tau^-$; say we have $w_{i_1 1} \in \tau^+$. Then, applying again Remark 2.4.10, we obtain $w_k, w_{ij} \in \tau^+$ for $k = 1, \ldots, m$, all i with $n_i \geq 3$ and $j = 1, \ldots, n_i$.

Assertion (iii) is proved analogously: treat first $\gamma_{i_1 1, i_1 2}$ with Lemma 2.4.9 (iii), then $\gamma_{i_1 1, ij}$ with Lemma 2.4.9 (iii) and (iv). Similarly, we obtain (iv) by treating first $\gamma_{i_1 1, i_2 1}$ and then all $\gamma_{i_1 1, ij}$ and $\gamma_{i_2 1, ij}$ with Lemma 2.4.9 (iii) and (iv). Finally, we obtain (v) using Lemma 2.4.9 (i). $\square$

Proposition 2.4.12. *Let $X = X(A, P, u)$ be non-toric, quasismooth and $\mathbb{Q}$-factorial with $\rho(X) = 2$. Assume that there is an elementary big cone in Σ_X and that we have $n_0 \geq \ldots \geq n_r$. If $m > 0$ holds, then there is a $\gamma_{ij,k} \in \mathrm{rlv}(u)$, we have $r = 2$ and the constellation of the n_i is $(n_0, 2, 2)$, $(2, 2, 1)$ or $(2, 1, 1)$.*

Proof. According to Lemma 2.4.11 (v), we may assume $w_1, \ldots, w_m \in \tau^+$. We claim that there is a $w_{i_1 j_1} \in \tau^-$ with $n_{i_1} \geq 2$. Otherwise, use Corollary 2.4.8 to see that there exist w_{ij} with $n_i \geq 2$ and Lemma 2.4.11 (i) to see that they all lie in τ^+. Since all monomials $T_i^{l_i}$ have the same degree in K, we obtain in addition $w_{i1} \in \tau^+$ for all i with $n_i = 1$. But then no weights w_{ij}, w_k are left to lie in τ^-, a contradiction.

Having verified the claim, we may take a $w_{i_1 j_1} \in \tau^-$ with $n_{i_1} \geq 2$. Then $\gamma_{i_1 j_1, 1} \in \mathrm{rlv}(u)$ is as desired. Moreover, Lemma 2.4.9 (ii) yields $r = 2$ and $n_{i_1} = 2$. If $n_0 \geq 3$ holds, then Lemma 2.4.11 (ii) gives $w_{ij} \in \tau^+$ for all i with $n_i \geq 3$. Moreover, as all $T_i^{l_i}$ share the same K-degree, we have $w_{i1} \in \tau^+$ for all i with

$n_i = 1$. By the same reason, one of the $w_{i_1 1}$, $w_{i_1 2}$ must lie in τ^+. As τ^- contains at least two weights, there is a $w_{i_2 j_2} \in \tau^-$ with $n_{i_2} = 2$ and $i_1 \neq i_2$. Thus, the constellation of $n_0 \geq n_1 \geq n_2$ is as claimed. □

Proposition 2.4.13. *Let $X = X(A, P, u)$ be non-toric, quasismooth and $\mathbb{Q}$-factorial with $\rho(X) = 2$. Assume that there is an elementary big cone in Σ_X and that we have $n_0 \geq \ldots \geq n_r$. If $m = 0$ holds, then there is a $\gamma_{i_1 j_1, i_2 j_2} \in \mathrm{rlv}(u)$, we have $r \leq 3$ and the constellation of the n_i is one of the following*

$$\begin{array}{ll} r = 2\colon & (n_0, 2, 2),\ (3, 2, 1),\ (3, 1, 1),\ (2, 2, 2),\ (2, 2, 1), \\ r = 3\colon & (2, 2, 2, 2),\ (2, 2, 2, 1),\ (2, 2, 1, 1). \end{array}$$

Proof. We first show $n_1 \leq 2$. Otherwise, we had $n_1 \geq 3$. Then, according to Lemma 2.4.11 (iv), we may assume that all the w_{ij} with $n_i \geq 3$ lie in τ^+. In particular, w_{11}, lies in τ^+. Because all monomials $T_i^{l_i}$ have the same degree in K, also $w_{i1} \in \tau^+$ holds for all i with $n_i = 1$. At least two weights $w_{i_1 j_1}$ and $w_{i_2 j_2}$ must belong to τ^-. For these, only $n_{i_1} = n_{i_2} = 2$ and $i_1 \neq i_2$ is possible. Applying Lemma 2.4.9 (iv) to $\gamma_{11, i_1 j_1} \in \mathrm{rlv}(u)$ gives $r = 2$, contradicting $n_0 \geq n_1 \geq 3$ and $n_{i_1} = n_{i_2} = 2$.

We treat the case $n_0 \geq 4$. By Lemma 2.4.11 (iii), we can assume $w_{01}, \ldots, w_{0n_0} \in \tau^+$. As before, we obtain $w_{i1} \in \tau^+$ for all i with $n_i = 1$ and we find two weights $w_{i_1 j_1}, w_{i_2 j_2} \in \tau^-$ with $n_{i_1} = n_{i_2} = 2$ and $i_1 \neq i_2$. Then $\gamma_{01, i_1 j_1} \in \mathrm{rlv}(u)$ is as wanted. Lemma 2.4.9 (iv) gives $r = 2$ and we end up with $(n_0, 2, 2)$.

Now consider the case $n_0 = 3$. Lemma 2.4.11 (i) guarantees that no w_{0j} lies in τ_X. If weights w_{0j} occur in both cones τ^+ and τ^-, say $w_{01} \in \tau^+$ and $w_{02} \in \tau^-$, then $\gamma_{01,02}$ is as wanted. Lemma 2.4.9 (iii) yields $r = 2$ and we obtain the constellations $(n_0, 2, 2)$, $(3, 2, 1)$ and $(3, 1, 1)$. So, assume that all weights w_{0j} lie in one of τ^+ and τ^-, say in τ^+. Then we proceed as in the case $n_0 \geq 4$ to obtain a $\gamma_{01, i_1 j_1} \in \mathrm{rlv}(u)$ and $r = 2$ with the constellation $(3, 2, 2)$.

Finally, consider the case $n_0 \leq 2$. Corollary 2.4.8 yields $n_0 = 2$. According to Lemma 2.4.11 (i) no w_{ij} with $n_i = 2$ lies in τ_X. So, we may assume $w_{01} \in \tau^+$. Moreover, all w_{ij} with $n_i = 1$ lie together in one τ^+, τ_X or in τ^-. Since each of τ^+ and τ^- contains two weights, we obtain $n_1 = 2$ and some $\gamma_{0j_1, 1j_2}$ is as wanted. Lemma 2.4.9 (iv) shows $r \leq 3$. □

We derive a special case of [**23**, Cor. 4.18].

Corollary 2.4.14. *Let $X = X(A, P, u)$ be smooth with $\rho(X) = 2$. Then the divisor class group $\mathrm{Cl}(X)$ is torsion-free.*

Proof. By Corollary 2.4.5, there is an elementary big cone in Σ_X. Thus, Propositions 2.4.12 and 2.4.13 deliver a two-dimensional $\gamma_0 \in \mathrm{rlv}(u)$. The corresponding weights generate K as a group. This gives $\mathrm{Cl}(X) \cong K \cong \mathbb{Z}^2$. □

Proof of Proposition 2.4.1. The variety X is isomorphic to some $X(A, P, u)$, where after suitable admissible operations we may assume $n_0 \geq \ldots \geq n_r$. Thus, Propositions 2.4.12 and 2.4.13 apply. □

2.5. Proof of Theorems 2.1.1, 2.1.2 and 2.1.4

We prove Theorems 2.1.1, 2.1.2 and 2.1.4 by going through the cases established in Proposition 2.4.1. The notation is the same as in Sections 1.4 and 2.4. We deal with a smooth projective variety $X = X(A, P, u)$ of Picard number $\rho(X) = 2$ coming with an effective torus action of complexity one.

From Corollary 2.4.14 we know that $\mathrm{Cl}(X) = K = \mathbb{Z}^2$ holds. With $w_{ij} = Q(e_{ij})$ and $w_k = Q(e_k)$, the columns of the $2 \times (n + m)$ degree matrix Q will be written as

$$w_{ij} \ = \ (w_{ij}^1, w_{ij}^2) \ \in \ \mathbb{Z}^2, \qquad\qquad w_k \ = \ (w_k^1, w_k^2) \ \in \ \mathbb{Z}^2.$$

Recall that all relations $g_0, \ldots, g_{r-2}$ of $R(A,P)$ have the same degree in $K = \mathbb{Z}^2$; we set for short

$$\mu \ = \ (\mu^1, \mu^2) \ := \ \deg(g_0) \ \in \ \mathbb{Z}^2.$$

We will frequently work with the faces of the orthant $\gamma = \mathbb{Q}^{n+m}_{\geq 0}$ introduced in Lemma 2.4.9:

$$\gamma_{ij,k} \ = \ \mathrm{cone}(e_{ij}, e_k) \ \preceq \ \gamma, \qquad \gamma_{i_1j_1,i_2j_2} \ = \ \mathrm{cone}(e_{i_1j_1}, e_{i_2j_2}) \ \preceq \ \gamma.$$

Remark 2.5.1. Consider $E \xrightarrow{Q} \mathbb{Z}^2$ and a face $\gamma_0 \preceq \gamma$ of type $\gamma_{ij,k}$, $\gamma_{i_1j_1,i_2j_2}$ or γ_{k_1,k_2}. Write e', e'' for the two generators of γ_0 and $w' = Q(e')$, $w'' = Q(e'')$ for the corresponding columns of the degree matrix Q such that (w', w'') is positively oriented in $\mathbb{Z}^2$. Then Remark 1.3.3 tells us

$$\gamma_0 \ \in \ \mathrm{rlv}(u) \quad \Rightarrow \quad \det(w', w'') \ = \ 1.$$

So, if $\gamma_0 \in \mathrm{rlv}(u)$ holds, then we may multiply Q from the left with a unimodular (2×2)-matrix transforming w' and w'' into $(1,0)$ and $(0,1)$. This change of coordinates on $\mathrm{Cl}(X)$ does not affect the defining data (A,P). If $w' = (1,0)$ and $w'' = (0,1)$ hold and $e \in \gamma$ is a canonical basis vector with corresponding column $w = Q(e)$, then we have

$$\begin{aligned} \mathrm{cone}(e', e) \in \mathrm{rlv}(u) \quad &\Rightarrow \quad w = (w^1, 1), \\ \mathrm{cone}(e'', e) \in \mathrm{rlv}(u) \quad &\Rightarrow \quad w = (1, w^2). \end{aligned}$$

We are ready to go through the cases of Proposition 2.4.1; we keep the numbering introduced there.

Case (I) (a). *We have $r = 2$, $m \geq 0$ and the list of n_i is $(n_0, 2, 2)$, where $n_0 \geq 3$. This leads to No. 1 and No. 2 in Theorems 2.1.1 and 2.1.2.*

Proof. In a first step we show that there occur weights w_{0j} in each of τ^+ and τ^-. Otherwise, we may assume that all w_{0j} lie in τ^+, see Lemma 2.4.11 (i). Then Lemma 2.4.11 (ii) says that also all w_k lie in τ^+. Moreover, we have $\deg(T_i^{l_i}) \in \tau^+$ for $i = 0,1,2$. Thus, we may assume $w_{11}, w_{21} \in \tau^+$ and obtain $w_{12}, w_{22} \in \tau^-$, as there must be at least two weights in τ^-. Finally, we may assume that $\mathrm{cone}(w_{01}, w_{12})$ contains $w_{02}, \ldots, w_{0n_0}$ and w_{22}. Applying Remark 2.5.1 first to $\gamma_{01,12}$, then to all $\gamma_{0j,12}$, $\gamma_{12,k}$ and $\gamma_{01,22}$, $\gamma_{12,21}$ yields

$$Q \ = \ \left[\begin{array}{cccc|cc|cc|ccc} 0 & w^1_{02} & \ldots & w^1_{0n_0} & w^1_{11} & 1 & w^1_{21} & 1 & w^1_1 & \ldots & w^1_m \\ 1 & 1 & \ldots & 1 & w^2_{11} & 0 & 1 & w^2_{22} & 1 & \ldots & 1 \end{array} \right],$$

where $w^1_{0j} \geq 0$ and $w^2_{22} \geq 0$. Since $\gamma_{01,12}, \gamma_{01,22} \in \mathrm{rlv}(u)$ holds, Lemma 2.4.9 (iv) implies $l_{11} = l_{21} = 1$. Applying $P \cdot Q^t = 0$ to the first row of P and the second row of Q gives

$$0 \ < \ 3 \ \leq \ n_0 \ \leq \ l_{01} + \ldots + l_{0n_0} \ = \ w^2_{11} \ = \ 1 + w^2_{22} w^1_{11},$$

where the last equality is due to $\gamma_{11,22} \in \mathrm{rlv}(u)$ and thus $\det(w_{22}, w_{11}) = 1$. We conclude $w^2_{22} > 0$ and $w^1_{11} > 0$. Because of $\gamma_{0j,22} \in \mathrm{rlv}(u)$, we obtain $\det(w_{22}, w_{0j}) = 1$. This implies $w^1_{0j} = 0$ for all $j = 2, \ldots, n_0$. Applying $P \cdot Q^t = 0$ to the first row of P and the first row of Q gives $w^1_{11} + l_{12} = 0$; a contradiction.

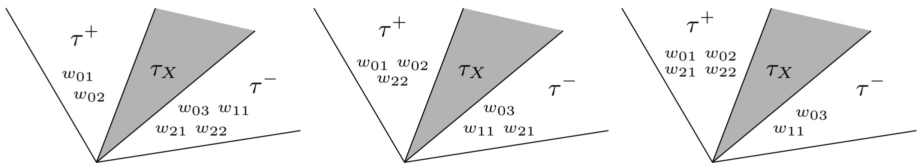
Knowing that each of τ^+ and τ^- contains weights w_{0j}, we can assume $w_{01}, w_{02} \in \tau^+$ and $w_{03} \in \tau^-$. Lemma 2.4.11 (ii) and (iii) show $n_0 = 3$ and $m = 0$. There is at least one other weight in τ^-, say $w_{11} \in \tau^-$. Applying Lemma 2.4.9 (iii) to $\gamma_{0j,03} \in \mathrm{rlv}(u)$ for $j = 1,2$ and (iv) to suitable $\gamma_{0j_1,i_2j_2} \in \mathrm{rlv}(u)$, we obtain

$$l_{01} = l_{02} = 1, \qquad l_{11} = l_{12} = 1, \qquad l_{21} = l_{22} = 1.$$

Moreover, Remark 2.5.1 applied to $\gamma_{01,03}$ as well as $\gamma_{02,03}$ and $\gamma_{01,11}$ brings the matrix Q into the shape

$$Q \;=\; \left[\begin{array}{ccc|cc|cc} 0 & w_{02}^1 & 1 & 1 & w_{12}^1 & w_{21}^1 & w_{22}^1 \\ 1 & 1 & 0 & w_{11}^2 & w_{12}^2 & w_{21}^2 & w_{22}^2 \end{array}\right].$$

Observe that the second component of the degree of the relation is $\mu^2 = 2$. The possible positions of the weights w_{2j} define three subcases:



(i) (ii) (iii)

We will see that cases (i) and (ii) give No. 1 and No. 2 of Theorem 2.1.1 respectively and case (iii) will not provide any smooth variety.

In (i) we assume $w_{21}, w_{22} \in \tau^-$. Then $\gamma_{01,21}, \gamma_{01,22} \in \mathrm{rlv}(u)$ holds and Remark 2.5.1 shows $w_{21}^1 = w_{22}^1 = 1$. This implies $\mu^1 = 2$. Similarly, considering $\gamma_{02,21}, \gamma_{02,22} \in \mathrm{rlv}(u)$, we obtain $w_{02}^1 = 0$ or $w_{21}^2 = w_{22}^2 = 0$. The latter contradicts $\mu^2 = 2$ and thus $w_{02}^1 = 0$ holds. We conclude $l_{03} = \mu^1 = 2$. Furthermore $w_{12}^1 = \mu^1 - w_{11}^1 = 1$. Together, we have

$$g_0 \;=\; T_{01}T_{02}T_{03}^2 + T_{11}T_{12} + T_{21}T_{22}, \qquad Q \;=\; \left[\begin{array}{ccc|cc|cc} 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & a & 2-a & b & 2-b \end{array}\right],$$

where $a, b \in \mathbb{Z}$. Observe that $w_{12} \in \tau^-$ must hold; otherwise, $\gamma_{03,12} \in \mathrm{rlv}(u)$ and Remark 2.5.1 yields $w_{12}^2 = 1$, contradicting $w_{12} = (1,1) = w_{11} \in \tau^-$. The semiample cone is $\mathrm{SAmple}(X) = \mathrm{cone}((0,1),(1,d))$, where $d = \max(a, 2-a, b, 2-b)$. The anticanonical class is $-\mathcal{K}_X = (3,4)$. Hence X is an almost Fano variety if and only if $d = 1$, which is equivalent to $a = b = 1$. In this situation X is already a Fano variety.

In (ii) we assume $w_{21} \in \tau^-$ and $w_{22} \in \tau^+$. Remark 2.5.1, applied to $\gamma_{01,21}, \gamma_{03,22} \in \mathrm{rlv}(u)$ shows $w_{21}^1 = w_{22}^2 = 1$. The latter implies $w_{21}^2 = \mu^2 - w_{22}^2 = 1$. We claim $w_{11}^2 \neq 0$. Otherwise, we have $w_{12}^2 = \mu^2 = 2$. This gives $\det(w_{03}, w_{12}) = 2$. We conclude $\gamma_{03,12} \notin \mathrm{rlv}(u)$ and $w_{12} \in \tau^-$. Then $\gamma_{01,12} \in \mathrm{rlv}(u)$ implies $w_{12}^1 = 1$. Thus, $w_{22} = (1,1)$ and $w_{12} = (1,2)$ hold, contradicting $w_{22} \in \tau^+$ and $w_{12} \in \tau^-$. Now, $\gamma_{11,22} \in \mathrm{rlv}(u)$ yields $w_{11}^2 w_{22}^1 = 0$ and thus $w_{22}^1 = 0$. We obtain $\mu^1 = 1$ and, as a consequence $l_{03} = 1, w_{02}^1 = 0$ and $w_{12}^1 = 0$. Therefore $w_{12} \in \tau^+$ holds. Now $\gamma_{03,12} \in \mathrm{rlv}(u)$ implies $w_{12}^2 = 1$ and $w_{11}^2 = \mu^2 - w_{12}^2 = 1$. We arrive at

$$g_0 \;=\; T_{01}T_{02}T_{03} + T_{11}T_{12} + T_{21}T_{22}, \qquad Q \;=\; \left[\begin{array}{ccc|cc|cc} 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 \end{array}\right].$$

The anticanonical class is $-\mathcal{K}_X = (2,4)$ and the semiample cone is $\mathrm{SAmple}(X) = \mathrm{cone}((0,1),(1,1))$. In particular X is Fano.

We turn to (iii), where both w_{21} and w_{22} lie in τ^+. The homogeneity of g_0 yields $w_{12} \in \tau^+$. Thus, $\gamma_{03,12}, \gamma_{03,21}, \gamma_{03,22} \in \mathrm{rlv}(u)$ holds and Remark 2.5.1 implies $w_{12}^2 = w_{21}^2 = w_{22}^2 = 1$. We conclude $w_{11}^2 = \mu^2 - w_{12}^2 = 1$. Similarly, $\gamma_{02,11}, \gamma_{11,21}, \gamma_{11,22} \in \mathrm{rlv}(u)$ yields $w_{02}^1 = w_{21}^1 = w_{22}^1 = 0$. This gives $0 \neq l_{03} = \mu^1 = w_{21}^1 + w_{22}^1 = 0$ which is not possible. $\square$

Case (I) (b). *We have $r = 2$, $m = 0$, $n = 6$ and the list of n_i is $(3,2,1)$. This leads to No.* 3 *in Theorems 2.1.1 and 2.1.2.*

Proof. Since there are at least two weights in τ^+ and another two in τ^-, we can assume $w_{01}, w_{02} \in \tau^+$ and $w_{03}, w_{12} \in \tau^-$. By Lemma 2.4.9 (iii) and (iv) we obtain

$l_{01} = l_{02} = l_{11} = l_{12} = 1$. We may assume that $\mathrm{cone}(w_{01}, w_{03})$ contains w_{02}. Applying Remark 2.5.1 firstly to $\gamma_{01,03}$, then to $\gamma_{02,03}$ and $\gamma_{01,12}$, we obtain

$$Q \;=\; \left[\begin{array}{ccc|cc|c} 0 & w_{02}^1 & 1 & w_{11}^1 & 1 & w_{21}^1 \\ 1 & 1 & 0 & w_{11}^2 & w_{12}^2 & w_{21}^2 \end{array}\right],$$

where $w_{02}^1 \ge 0$. For the degree μ of g_0, we have $\mu^2 = 2$. We conclude $w_{11}^2 = 2 - w_{12}^2$ and $l_{21}w_{21}^2 = 2$ which in turn implies $l_{21} = 2$ and $w_{21}^2 = 1$. For $\gamma_{02,12} \in \mathrm{rlv}(u)$, Remark 2.5.1 gives $\det(w_{12}, w_{02}) = 1$ and thus $w_{02}^1 = 0$ or $w_{12}^2 = 0$ must hold.

We treat the case $w_{02}^1 = 0$. Then $\mu = (l_{03}, 2)$ holds. We conclude $w_{11}^1 = l_{03} - 1$ and $w_{21}^1 = l_{03}/2$. With $c := l_{03}/2 \in \mathbb{Z}_{\ge 1}$ and $a := w_{12}^2 \in \mathbb{Z}$, we obtain the degree matrix

$$Q \;=\; \left[\begin{array}{ccc|cc|c} 0 & 0 & 1 & 2c-1 & 1 & c \\ 1 & 1 & 0 & 2-a & a & 1 \end{array}\right].$$

We show $w_{11} \in \tau^-$. Otherwise, $w_{11} \in \tau^+$ holds, we have $\gamma_{03,11} \in \mathrm{rlv}(u)$ and Remark 2.5.1 yields $a = 1$. But then $w_{01} = (0,1) \in \tau^+$ and $w_{11} = (2c-1, 1) \in \tau^+$ imply $w_{12} = (1,1) \in \tau^+$; a contradiction. So we have $w_{11} \in \tau^-$. Then $\gamma_{01,11} \in \mathrm{rlv}(u)$ holds. Remark 2.5.1 gives $\det(w_{11}, w_{01}) = 1$ which means $c = 1$ and, as a consequence, $l_{03} = 2$. Together, we have

$$g_0 \;=\; T_{01}T_{02}T_{03}^2 + T_{11}T_{12} + T_{21}^2, \qquad Q \;=\; \left[\begin{array}{ccc|cc|c} 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 2-a & a & 1 \end{array}\right],$$

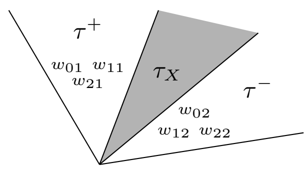
where we may assume $a \ge 2 - a$ that means $a \in \mathbb{Z}_{\ge 1}$. The semiample cone is $\mathrm{SAmple}(X) = \mathrm{cone}((0,1),(1,a))$, and the anticanonical class is $-\mathcal{K}_X = (2,3)$. In particular, X is an almost Fano variety if and only $a = 1$ holds. In this situation X is already a Fano variety.

We turn to the case $w_{12}^2 = 0$. Here, $w_{11}^2 = \mu^2 = 2$ leads to $\det(w_{03}, w_{11}) = 2$ and thus the $\mathfrak{F}$-face $\gamma_{03,11}$ does not belong to $\mathrm{rlv}(u)$; see Remark 2.5.1. Hence $w_{11} \in \tau^-$ and thus $\gamma_{01,11} \in \mathrm{rlv}(u)$. This gives $w_{11}^1 = 1$ and thus $w_{11} = (1,2)$. Because of $w_{02} = (w_{02}, 1) \in \tau^+$, we must have $w_{02}^1 = 0$ and the previous consideration applies. □

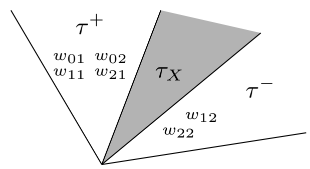
Case (I) (c). *We have $r = 2$, $m = 0$, $n = 5$ and the list of n_i is $(3,1,1)$. This case does not provide smooth varieties.*

Proof. Each of τ^+ and τ^- contains at least two weights. We may assume $w_{01}, w_{02} \in \tau^+$ and $w_{03}, w_{11}, w_{21} \in \tau^-$. Then $\gamma_{01,03}, \gamma_{02,03} \in \mathrm{rlv}(u)$ holds and Lemma 2.4.9 (iii) yields $l_{01} = l_{02} = 1$. By Remark 2.5.1 we can assume $w_{03} = (1,0)$ and $w_{01}^2 = w_{02}^2 = 1$. This implies $\mu^2 = 2$ and, as a consequence, $l_{11} = l_{21} = 2$. By [**36**, Thm. 1.1], we have torsion in $\mathrm{Cl}(X)$; a contradiction to Corollary 2.4.14. □

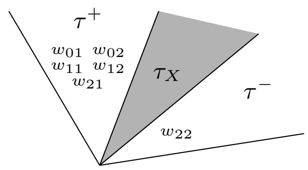
Case (I) (d). *We have $r = 2$, $m \geq 0$, $n = 6$ and the list of n_i is $(2, 2, 2)$. Suitable admissible operations lead to one of the following configurations for the weights w_{ij}:*



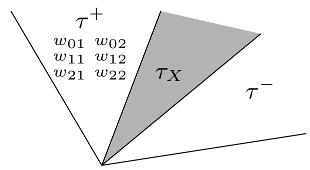
(i)



(ii)



(iii)



(iv)

Configuration (i) amounts to No. 4 in Theorems 2.1.1, 2.1.2 and 2.1.4, configuration (ii) to No. 5, configuration (iii) to Nos. 6 and 7, and configuration (iv) to Nos. 8 and 9.

Proof for configuration (i). We have $w_{01}, w_{11}, w_{21} \in \tau^+$ and $w_{02}, w_{12}, w_{22} \in \tau^-$. We may assume $w_k \in \tau^+$ for all $k = 1, \ldots, m$. If $m > 0$, we have $\gamma_{i2,1} \in \operatorname{rlv}(u)$ and Lemma 2.4.9 (ii) gives $l_{i1} = 1$ for $i = 0, 1, 2$. If $m = 0$, we use $\gamma_{i_1 1, i_2 2} \in \operatorname{rlv}(u)$ and Lemma 2.4.9 (iv) to obtain $l_{i_1 2} = 1$ or $l_{i_2 1} = 1$ for all $i_1 \neq i_2$. Thus, for $m = 0$, we may assume $l_{01} = l_{11} = 1$ and are left with $l_{21} = 1$ or $l_{22} = 1$.

We treat the case $m \geq 0$ and $l_{01} = l_{11} = l_{21} = 1$. Here we may assume $w_{11}, w_{21}, w_{22} \in \operatorname{cone}(w_{01}, w_{12})$. Applying Remark 2.5.1 firstly to $\gamma_{01,12}$ and then to $\gamma_{01,22}$, $\gamma_{12,21}$ and all $\gamma_{12,k}$ gives

$$Q \ = \ \left[\begin{array}{cc|cc|cc|ccc} 0 & w_{02}^1 & w_{11}^1 & 1 & w_{21}^1 & 1 & w_1^1 & \ldots & w_m^1 \\ 1 & w_{02}^2 & w_{11}^2 & 0 & 1 & w_{22}^2 & 1 & \ldots & 1 \end{array}\right].$$

Using $w_{11}, w_{21}, w_{22} \in \operatorname{cone}(w_{01}, w_{12})$ and the fact that the determinants of (w_{02}, w_{01}), (w_{12}, w_{11}) and (w_{22}, w_{21}) are positive, we obtain

$$w_{11}^1,\, w_{21}^1,\, w_{22}^2 \ \geq \ 0, \qquad w_{02}^1,\, w_{11}^2 \ > \ 0, \qquad 1 \ > \ w_{22}^2 w_{21}^1.$$

The degree μ of the relation satisfies

$$0 \ < \ \mu^1 \ = \ l_{02} w_{02}^1 \ = \ w_{11}^1 + l_{12} \ = \ w_{21}^1 + l_{22},$$

$$0 \ < \ \mu^2 \ = \ 1 + l_{02} w_{02}^2 \ = \ w_{11}^2 \ = \ 1 + l_{22} w_{22}^2.$$

In particular, $w_{02}^2 \geq 0$ holds and thus all components of the w_{ij} are non-negative. With $\gamma_{02,11}, \gamma_{02,21}, \in \operatorname{rlv}(u)$ and Remark 2.5.1, we obtain

$$w_{02}^1 w_{11}^2 \ = \ 1 + w_{02}^2 w_{11}^1, \qquad\qquad w_{02}^1 - 1 \ = \ w_{02}^2 w_{21}^1.$$

We show $w_{22}^2 = 0$. Otherwise, because of $1 > w_{22}^2 w_{21}^1$, we have $w_{21}^1 = 0$. This implies $w_{02}^1 = 1$ and thus

$$w_{11}^2 \ = \ 1 + w_{02}^2 w_{11}^1 \ = \ 1 + l_{02} w_{02}^2.$$

This gives $w_{02}^2 = 0$ or $w_{11}^1 = l_{02}$. The first is impossible because of $l_{02} w_{02}^2 = l_{22} w_{22}^2$ and the second because of $l_{02} = l_{02} w_{02}^1 = w_{11}^1 + l_{12}$.

Knowing $w_{22}^2 = 0$, we directly conclude $w_{11}^2 = 1$ and $w_{02}^2 = 0$ from $\mu^2 = 1$. This gives $w_{02}^1 = 1$. With $a := w_{11}^1 \in \mathbb{Z}_{\geq 0}$, $b := w_{21}^1 \in \mathbb{Z}_{\geq 0}$ and $c_k := w_k^1 \in \mathbb{Z}$ we are in the situation

$$g_0 \;=\; T_{01}T_{02}^{l_{02}}+T_{11}T_{12}^{l_{12}}+T_{21}T_{22}^{l_{22}}, \qquad Q \;=\; \left[\begin{array}{cc|cc|cc|ccc} 0 & 1 & a & 1 & b & 1 & c_1 & \dots & c_m \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & \dots & 1 \end{array}\right],$$

where we may assume $0 \le a \le b$ and $c_1 \le \ldots \le c_m$. Observe $l_{02} = a + l_{12} = b + l_{22}$. The anticanonical class and the semiample cone of X are given by

$$\begin{aligned} -\mathcal{K}_X &= (3 + b + c_1 + \ldots + c_m - l_{12},\, 2 + m), \\ \mathrm{SAmple}(X) &= \mathrm{cone}((1, 0), (d, 1)), \end{aligned}$$

where $d := \max(b, c_m)$. Consequently, X is a Fano variety if and only if the following inequality holds

$$3 + b + c_1 + \ldots + c_m - l_{12} \;>\; (2 + m)d.$$

A necessary condition for this is $0 \le d \le 1$ with $l_{12} = 1$ if $d = 1$ and $l_{12} \le 2$ if $d = 0$ The tuples $(a, b, d, l_{02}, l_{12}, l_{22})$ fulfilling that condition are

$$(0, 0, 0, 2, 2, 2), \qquad (0, 0, 0, 1, 1, 1), \qquad (1, 1, 1, 2, 1, 1).$$

Each of these three tuples leads indeed to a Fano variety X; the respectively possible choices of the c_k lead to Nos. 4.A, 4.B and 4.C of Theorem 2.1.2 and are as follows:

$$c_1 = \ldots = c_m = 0, \qquad -1 \le c_1 \le 0 = c_2 = \ldots = c_m, \qquad c_1 = \ldots = c_m = 1.$$

Moreover X is a truly almost Fano variety if and only if the following equality holds

$$3 + b + c_1 + \ldots + c_m - l_{12} \;=\; (2 + m)d.$$

This implies $0 \le d \le 2$ and the only possible parameters fulfilling that condition are listed as Nos. 4.A to 4.F in the table of Theorem 2.1.4.

We turn to the case $m = 0$, $l_{01} = l_{11} = 1$ and $l_{21} \ge 2$. Lemma 2.4.9 (iv) applied to $\gamma_{01,22}, \gamma_{11,22} \in \mathrm{rlv}(u)$ gives $l_{02} = l_{12} = 1$. If $l_{22} = 1$ holds, then suitable admissible operations bring us to the previous case. Hence consider $l_{22} \ge 2$. We may assume $w_{11} \in \mathrm{cone}(w_{01}, w_{12})$. We apply Remark 2.5.1 firstly to $\gamma_{01,12}$, then to $\gamma_{01,22}$, $\gamma_{12,21}$ and arrive at

$$g_0 \;=\; T_{01}T_{02} + T_{11}T_{12} + T_{21}^{l_{21}}T_{22}^{l_{22}}, \quad Q \;=\; \left[\begin{array}{cc|cc|cc} 0 & w_{02}^1 & w_{11}^1 & 1 & w_{21}^1 & 1 \\ 1 & w_{02}^2 & w_{11}^2 & 0 & 1 & w_{22}^2 \end{array}\right],$$

where $w_{11}^1 \ge 0$ and $w_{11}^2 = \det(w_{12}, w_{11}) > 0$. We have $\mu = w_{02} + w_{01} = w_{11} + w_{12}$ and thus $w_{02} = w_{11} + w_{12} - w_{01}$. Because of $\gamma_{02,11} \in \mathrm{rlv}(u)$, we obtain

$$1 \;=\; \det(w_{02}, w_{11}) \;=\; \det(w_{12} - w_{01}, w_{11}) \;=\; w_{11}^1 + w_{11}^2.$$

We conclude $w_{11} = (0, 1)$ and $\mu = (1, 1)$. Using $\mu = l_{21}w_{21} + l_{22}w_{22}$ and $l_{21}, l_{22} \ge 2$ we see $w_{21}^1, w_{22}^2 < 0$. On the other hand, $0 < \det(w_{22}, w_{21}) = 1 - w_{21}^1 w_{22}^2$, a contradiction. Thus $l_{22} \ge 2$ does not occur. □

Proof for configuration (ii). We have $w_{01}, w_{02}, w_{11}, w_{21} \in \tau^+$ and $w_{12}, w_{22} \in \tau^-$. We may assume that $w_{02}, w_{12} \in \mathrm{cone}(w_{01}, w_{22})$ holds. Applying Remark 2.5.1 first to $\gamma_{01,22} \in \mathrm{rlv}(u)$ and then to $\gamma_{01,12}, \gamma_{02,22}, \gamma_{11,22} \in \mathrm{rlv}(u)$ we obtain

$$Q \;=\; \left[\begin{array}{cc|cc|cc|ccc} 0 & w_{02}^1 & w_{11}^1 & 1 & w_{21}^1 & 1 & w_1^1 & \dots & w_m^1 \\ 1 & 1 & 1 & w_{12}^2 & w_{21}^2 & 0 & w_1^2 & \dots & w_m^2 \end{array}\right],$$

where we have $w_{02}^1, w_{12}^2 \ge 0$ due to $w_{02}, w_{12} \in \mathrm{cone}(w_{01}, w_{22})$. Moreover, $w_{21}^2 > 0$ holds, as we infer from the conditions

$$\begin{aligned} 0 \;\le\; \mu^1 \;&=\; l_{02}w_{02}^1 \;=\; l_{11}w_{11}^1 + l_{12} \;=\; l_{21}w_{21}^1 + l_{22}, \\ 0 \;<\; \mu^2 \;&=\; l_{01} + l_{02} \;=\; l_{11} + l_{12}w_{12}^2 \;=\; l_{21}w_{21}^2. \end{aligned}$$

We show $l_{11} \ge 2$. Otherwise, the above conditions give $l_{12}w_{12}^2 > 0$ and thus $w_{12}^2 > 0$. For $\gamma_{02,12} \in \mathrm{rlv}(u)$, Remark 2.5.1 gives $\det(w_{12}, w_{02}) = 1$ which means $w_{12}^2 w_{02}^1 = 0$ and thus $w_{02}^1 = 0$. This implies $l_{21}w_{21}^1 + l_{22} = 0$ and thus $w_{21}^1 < 0$;

a contradiction to $1 = \det(w_{12}, w_{21}) = w_{21}^2 - w_{12}^2 w_{21}^1$ which in turn holds due to $\gamma_{12,21} \in \mathrm{rlv}(u)$ and Remark 2.5.1.

Lemma 2.4.9 (iv) applied to $\gamma_{02,12}, \gamma_{01,12}, \gamma_{21,12} \in \mathrm{rlv}(u)$ shows $l_{01} = l_{02} = l_{22} = 1$. Putting together $\mu^2 = 2 = l_{11} + l_{12} w_{12}^2$ and $l_{11} \neq 1$, we conclude $l_{11} = 2$ and $w_{12}^2 = 0$. With $\gamma_{12,21} \in \mathrm{rlv}(u)$ and Remark 2.5.1 we obtain $w_{21}^2 = 1$ and hence $l_{21} = \mu^2 = 2$. From

$$0 \;\le\; \mu^1 \;=\; w_{02}^1 \;=\; 2w_{11}^1 + 1 \;=\; 2w_{21}^1 + 1$$

we conclude $w_{11}^1 = w_{21}^1 \ge 0$ and thus $w_{02}^1 > 0$. Lemma 2.4.9 (ii) implies that possible weights of type w_k lie in τ^-. Thus Remark 2.5.1 and $\gamma_{01,k}$ imply $w_k^1 = 1$ for all k. Moreover, since $\gamma_{02,k} \in \mathrm{rlv}(u)$, the latter implies $w_k^2 = 0$. All in all, we arrive at

$$g_0 \;=\; T_{01}T_{02} + T_{11}^2 T_{12} + T_{21}^2 T_{22}, \qquad Q \;=\; \left[\begin{array}{cc|cc|cc|ccc} 0 & 2a+1 & a & 1 & a & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & \dots & 0 \end{array}\right],$$

where $a \in \mathbb{Z}_{\ge 0}$. The anticanonical class is $-\mathcal{K}_X = (2a+2+m, 2)$ and the semiample cone is $\mathrm{SAmple}(X) = \mathrm{cone}((1,0),(2a+1,1))$. Hence X is an almost Fano variety if and only if $m \ge 2a$ holds and X is a Fano variety if and only if $m > 2a$ holds. □

Proof for configuration (iii). We have $w_{01}, w_{02}, w_{11}, w_{12}, w_{21} \in \tau^+$ and $w_{22} \in \tau^-$. As there must be another weight in τ^-, we obtain $m > 0$. Lemma 2.4.11 (v) yields $w_1, \dots, w_m \in \tau^-$. We may assume $w_{02}, w_{11}, w_{12}, w_k \in \mathrm{cone}(w_{01}, w_1)$, where $k = 2, \dots, m$. Applying Remark 2.5.1 firstly to $\gamma_{01,1} \in \mathrm{rlv}(u)$ and then to the remaining faces $\gamma_{01,22}, \gamma_{01,k}, \gamma_{ij,1}$ from $\mathrm{rlv}(u)$ leads to the degree matrix

$$Q \;=\; \left[\begin{array}{cc|cc|cc|cccc} 0 & w_{02}^1 & w_{11}^1 & w_{12}^1 & w_{21}^1 & 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 1 & 1 & w_{22}^2 & 0 & w_2^2 & \dots & w_m^2 \end{array}\right]$$

with at most w_{21}^1, w_{22}^2 negative. We infer $l_{01} = l_{02} = l_{11} = l_{12} = l_{22} = 1$ from Lemma 2.4.9 (ii). For $\gamma_{02,22}, \gamma_{11,22}, \gamma_{12,22} \in \mathrm{rlv}(u)$ Remark 2.5.1 tells us

$$w_{22}^2 \;=\; 0 \qquad \text{or} \qquad w_{02}^1 \;=\; w_{11}^1 \;=\; w_{12}^1 \;=\; 0.$$

We treat the case $w_{22}^2 = 0$. Here $l_{21} = \mu^2 = 2$ holds. Thus $\mu^1 = w_{02}^1 = 2w_{21}^1 + 1$ holds. Because of $w_{02}^1 \ge 0$, we conclude $w_{02}^1 > 0$ and $w_{21}^1 \ge 0$. Remark 2.5.1 applied to $\gamma_{02,k} \in \mathrm{rlv}(u)$ gives $w_k^2 = 0$ for all $k = 2, \dots, m$. We arrive at

$$g_0 \;=\; T_{01}T_{02} + T_{11}T_{12} + T_{21}^2 T_{22}, \qquad Q \;=\; \left[\begin{array}{cc|cc|cc|ccc} 0 & 2c+1 & a & b & c & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & \dots & 0 \end{array}\right],$$

where $a, b, c \in \mathbb{Z}_{\ge 0}$ and $a + b = 2c + 1$. Furthermore, the anticanonical class is $-\mathcal{K}_X = (3c + 2 + m, 3)$ and we have $\mathrm{SAmple}(X) = \mathrm{cone}((1,0),(2c+1,1))$. In particular, X is an almost Fano variety if and only if $3c + 1 \le m$ holds and a Fano variety if and only if the corresponding strict inequality holds.

Now we consider the case $w_{02}^1 = w_{11}^1 = w_{12}^1 = 0$. We have $\mu^1 = 0$, which implies $l_{21} = 1$, $w_{21}^1 = -1$. Consequently, $\mu^2 = 2$ gives $w_{22}^2 = 1$. Since $\gamma_{21,k} \in \mathrm{rlv}(u)$ for $2 \le k \le m$, we conclude $w_k^2 = 0$ for all k. Therefore we obtain

$$g_0 \;=\; T_{01}T_{02} + T_{11}T_{12} + T_{21}T_{22}, \qquad Q \;=\; \left[\begin{array}{cc|cc|cc|ccc} 0 & 0 & 0 & 0 & -1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & \dots & 0 \end{array}\right].$$

Finally, we have $-\mathcal{K}_X = (m, 4)$ and $\mathrm{SAmple}(X) = \mathrm{cone}((1,1),(0,1))$. Thus, X is a Fano variety if and only if $m < 4$ holds. Moreover, X is an almost Fano variety if and only if $m \le 4$ holds. □

Proof for configuration (iv). All w_{ij} lie in τ^+. Then we have $m \ge 2$ and one and hence all w_k in lie in τ^-, see Lemma 2.4.11 (v). Applying Lemma 2.4.9 (ii) to $\gamma_{ij,1} \in \mathrm{rlv}(u)$, we conclude $l_{ij} = 1$ for all i, j. Thus we have the relation

$$g_0 \;=\; T_{01}T_{02} + T_{11}T_{12} + T_{21}T_{22}.$$

We may assume that $\operatorname{cone}(w_{01}, w_1)$ contains all w_{ij}, w_k. Remark 2.5.1 applied to $\gamma_{01,1} \in \operatorname{rlv}(u)$ leads to $w_1 = (1,0)$ and $w_{01} = (0,1)$. All other weights lie in the positive orthant. For $\gamma_{ij,1}, \gamma_{01,k} \in \operatorname{rlv}(u)$ Remark 2.5.1 shows $w_{ij}^2 = w_k^1 = 1$ for all i, j, k. Consider the case that all w_k^2 vanish. Then the degree matrix is of the form

$$Q \;=\; \left[\begin{array}{cc|cc|cc|ccc} 0 & a_2 & a_3 & a_4 & a_5 & a_6 & 1 & \dots & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & \dots & 0 \end{array}\right],$$

where $a_i \in \mathbb{Z}_{\geq 0}$ and $a_2 = a_3 + a_4 = a_5 + a_6$. We have $-\mathcal{K}_X = (2a_2 + m, 4)$ and $\operatorname{SAmple}(X) = \operatorname{cone}((1,0),(a_2,1))$. Hence X is a Fano variety if and only if $2a_2 < m$ holds and an almost Fano variety if and only if $2a_2 \leq m$ holds.

Finally, let w_k^2 be strictly positive for some k. Note that we may assume $0 \leq w_2^2 \leq \ldots \leq w_m^2$; in particular $w_m^2 > 0$. Since $\gamma_{ij,m} \in \operatorname{rlv}(u)$ for all i, j, Remark 2.5.1 yields $w_{ij}^1 = 0$ for all i, j. Thus we obtain the degree matrix

$$Q \;=\; \left[\begin{array}{cc|cc|cc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & a_2 & \dots & a_m \end{array}\right],$$

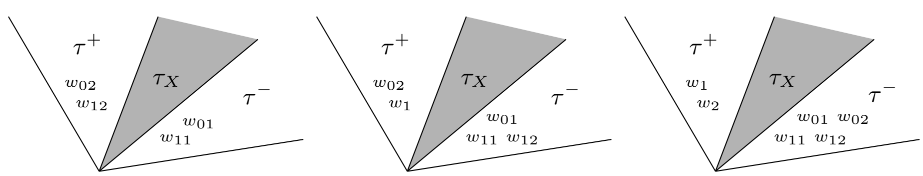
where $0 \leq a_2 \leq \ldots \leq a_m$ and $a_m > 0$. The anticanonical class and the semiample cone are given as

$$-\mathcal{K}_X \;=\; (m,\, 4 + a_2 + \ldots + a_m), \qquad \operatorname{SAmple}(X) \;=\; \operatorname{cone}((0,1),(1,a_m)).$$

In particular, X is a Fano variety if and only if $4 + a_2 + \ldots + a_m > m a_m$ holds. Note that for the latter $a_m \leq 3$ is necessary. Moreover, X is a truly almost Fano variety if and only if the equality $4 + a_2 + \ldots + a_m = m a_m$ holds. $\square$

Case (I) (e). *We have $r = 2$, $m \geq 0$, $n = 5$ and the list of n_i is $(2,2,1)$. This leads to Nos. 10, 11 and 12 in Theorems 2.1.1, 2.1.2 and 2.1.4.*

Proof. We divide this case into the following three configurations, according to the way some weights lie with respect to τ_X.



(i) (ii) (iii)

We show that configuration (i) does not provide any smooth variety, (ii) delivers No. 10 of Theorem 2.1.1 and (iii) delivers Nos. 11 and 12.

In configuration (i) we have $w_{01}, w_{11} \in \tau^-$ and $w_{02}, w_{12} \in \tau^+$. We may assume $w_{11} \in \operatorname{cone}(w_{01}, w_{12})$. Remark 2.5.1 applied to $\gamma_{01,12} \in \operatorname{rlv}(u)$ leads to $w_{01} = (1,0)$ and $w_{12} = (0,1)$. Observe $w_{11}^1, w_{11}^2 \geq 0$. Due to $\det(w_{11}, w_{12}) > 0$, we even have $w_{11}^1 > 0$ and $\det(w_{01}, w_{02}) > 0$ gives $w_{02}^2 > 0$. Since $T_0^{l_0}$ and $T_1^{l_1}$ share the same degree, we have

$$l_{01} w_{01} + l_{02} w_{02} \;=\; l_{11} w_{11} + l_{12} w_{12}.$$

Lemma 2.4.9 (iv) says $l_{02} = 1$ or $l_{11} = 1$, which allows us to resolve for w_{02} or for w_{11} in the above equation. Using $\gamma_{02,11} \in \operatorname{rlv}(u)$, we obtain

$$\begin{aligned} l_{02} = 1 \quad &\Longrightarrow \quad 1 = \det(w_{11}, w_{02}) = \det(w_{11}, l_{12} w_{12} - l_{01} w_{01}) = l_{12} w_{11}^1 + l_{01} w_{11}^2, \\ l_{11} = 1 \quad &\Longrightarrow \quad 1 = \det(w_{11}, w_{02}) = \det(l_{01} w_{01} - l_{12} w_{12}, w_{02}) = l_{01} w_{02}^2 + l_{12} w_{02}^1. \end{aligned}$$

We show $l_{02} > 1$. Otherwise, $l_{02} = 1$ holds. The above consideration shows $w_{11}^2 = 0$ and $l_{12} = w_{11}^1 = 1$. Thus, $l_{21} w_{21}^2 = l_{12} = 1$ holds and we obtain $l_{21} = 1$; a contradiction to P being irredundant. Thus, $l_{02} > 1$ and $l_{11} = 1$ must hold. Because of $w_{02}^2 > 0$, we must have $w_{02}^1 \leq 0$. With

$$1 \;=\; \det(w_{11}, w_{02}) \;=\; w_{11}^1 w_{02}^2 - w_{11}^2 w_{02}^1$$

we see $w_{11}^2 w_{02}^1 = 0$ and $w_{11}^1 = w_{02}^2 = 1$. But then we arrive at $1 = l_{11}w_{11}^1 = l_{21}w_{21}^1$. Again this means $l_{21} = 1$; a contradiction to P being irredundant.

In configuration (ii) we have $w_{01}, w_{11}, w_{12} \in \tau^-$ and $w_{02}, w_1 \in \tau^+$. In particular $m \geq 1$. Lemma 2.4.11 (v) yields $w_2, \ldots, w_m \in \tau^+$. Applying Remark 2.5.1 first to $\gamma_{11,1} \in \mathrm{rlv}(u)$ an then to $\gamma_{01,1}, \gamma_{12,1}, \gamma_{02,11}, \gamma_{11,k} \in \mathrm{rlv}(u)$ leads to

$$Q \; = \; \left[\begin{array}{cc|cc|c|cccc} 1 & w_{02}^1 & 1 & 1 & w_{21}^1 & 0 & w_2^1 & \ldots & w_m^1 \\ w_{01}^2 & 1 & 0 & w_{12}^2 & w_{21}^2 & 1 & 1 & \ldots & 1 \end{array} \right].$$

Applying Lemma 2.4.9 (ii) to $\gamma_{01,1}, \gamma_{12,1}, \gamma_{11,1} \in \mathrm{rlv}(u)$ we obtain $l_{02} = l_{11} = l_{12} = 1$. For the degree μ of the relation g_0 we note

$$\mu^1 \; = \; l_{01} + w_{02}^1 \; = \; 2 \; = \; l_{21}w_{21}^1, \qquad\qquad \mu^2 \; = \; l_{01}w_{01}^2 + 1 \; = \; w_{12}^2 \; = \; l_{21}w_{21}^2.$$

From $\mu^1 = 2$ we infer $l_{21} = 2$ and $w_{21}^1 = 1$. Consequently, μ^2 is even and both l_{01}, w_{01}^2 are odd. Using again $\mu^1 = 2$ gives $w_{02}^1 \neq 0$. For $\gamma_{02,12} \in \mathrm{rlv}(u)$ Remark 2.5.1 yields $\det(w_{12}, w_{02}) = 1$ which means $w_{02}^1 w_{12}^2 = 0$. We conclude $w_{12}^2 = 0 = \mu^2$. This implies $w_{21}^2 = 0$, $w_{01}^2 = -1$, $l_{01} = 1$ and $w_{02}^1 = 1$. We obtain

$$g_0 \; = \; T_{01}T_{02} + T_{11}T_{12} + T_{21}^2, \qquad Q \; = \; \left[\begin{array}{cc|cc|c|ccc} 1 & 1 & 1 & 1 & 1 & 0 & \ldots & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 & \ldots & 1 \end{array} \right],$$

where $w_2^1 = \ldots = w_m^1 = 0$ follows from Remark 2.5.1 applied to $\gamma_{01,k} \in \mathrm{rlv}(u)$. The semiample cone is given as $\mathrm{SAmple}(X) = \mathrm{cone}((1,0),(1,1))$ and the anticanonical class as $-\mathcal{K}_X = (3, m)$. Therefore X is a Fano variety if and only if $m < 3$, i.e $m = 1, 2$. Moreover, X is an almost Fano variety if and only if $m \leq 3$.

In configuration (iii) we have $w_{01}, w_{02}, w_{11}, w_{12} \in \tau^-$ and $w_1, w_2 \in \tau^+$. In particular $m \geq 2$. Lemma 2.4.11 (v) ensures $w_3, \ldots, w_m \in \tau^+$. We can assume that all w_{ij}, w_k lie in $\mathrm{cone}(w_{01}, w_1)$. Applying Remark 2.5.1, firstly to $\gamma_{01,1}$ and then to all relevant faces of the types $\gamma_{ij,1}$ and $\gamma_{01,k}$, we achieve

$$w_{01} = (1,0), \quad w_1 = (0,1), \quad w_{02}^1 = w_{11}^1 = w_{12}^1 = 1, \quad w_2^2 = \ldots = w_m^2 = 1.$$

Lemma 2.4.9 (ii) applied to all $\gamma_{ij,1}$ shows $l_{ij} = 1$ for all i, j. We conclude $\mu^1 = 2$ which in turn implies $l_{21} = 2$ and $w_{21}^1 = 1$. In particular, we have the relation

$$g_0 \; = \; T_{01}T_{02} + T_{11}T_{12} + T_{21}^2.$$

We treat the case that $w_1^1 = \ldots = w_m^1 = 0$ holds. All columns of the degree matrix lie in $\mathrm{cone}(w_{01}, w_1)$ and thus Q is of the form

$$Q \; = \; \left[\begin{array}{cc|cc|c|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 2c & a & b & c & 1 & 1 & \ldots & 1 \end{array} \right],$$

where $a, b, c \in \mathbb{Z}_{\geq 0}$ and $a + b = 2c$. The anticanonical class is $-\mathcal{K} = (3, m + 3c)$ and we have $\mathrm{SAmple}(X) = \mathrm{cone}((0,1),(1,2c))$. Therefore X is a Fano variety if and only if $m > 3c$. Moreover, X is an almost Fano variety if and only if $m \geq 3c$.

We treat the case that $w_k^1 > 0$ holds for some k. Then we obtain $w_{02}^2 = 0$ by applying Remark 2.5.1 to $\gamma_{02,k}$. This yields $\mu^2 = 0$ and thus $w_{ij}^2 = 0$ for all i, j. Consequently, the degree matrix is given as

$$Q \; = \; \left[\begin{array}{cc|cc|c|cccc} 1 & 1 & 1 & 1 & 1 & 0 & w_2^1 & \ldots & w_m^1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & \ldots & 1 \end{array} \right],$$

where we can assume $0 \leq w_2^1 \leq \ldots \leq w_m^1$. The semiample cone and the anticanonical divisor are given as

$$\mathrm{SAmple}(X) = \mathrm{cone}((1,0),(w_m^1, 1)), \qquad -\mathcal{K} = (3 + w_2^1 + \ldots + w_m^1, m).$$

We see that X is an almost Fano variety if and only if $mw_m^1 \leq 3 + w_2^1 + \ldots + w_m^1$ and that X is a Fano variety if and only if the corresponding strict inequality holds. $\square$

Case (I) (f). *We have $r = 2$, $m \geq 1$, $n = 4$ and the list of n_i is $(2,1,1)$. This case does not provide any smooth variety.*

Proof. We can assume $w_{01} \in \tau^-$ and $w_1 \in \tau^+$. Lemma 2.4.11 (v) ensures $w_2, \ldots, w_m \in \tau^+$. Applying Remark 2.5.1 first to $\gamma_{01,1} \in \mathrm{rlv}(u)$ and then to the remaining $\gamma_{01,k} \in \mathrm{rlv}(u)$, we achieve

$$Q \;=\; \left[\begin{array}{cc|c|c|cccc} 1 & w_{02}^1 & w_{11}^1 & w_{21}^1 & 0 & w_2^1 & \ldots & w_m^1 \\ 0 & w_{02}^2 & w_{11}^2 & w_{21}^2 & 1 & 1 & \ldots & 1 \end{array}\right].$$

Moreover $\gamma_{01,1} \in \mathrm{rlv}(u)$ implies $l_{02} = 1$ by Lemma 2.4.9 (ii). Recall from Corollary 2.4.14 that $\mathrm{Cl}(X)$ is torsion-free. Thus [**36**, Thm. 1.1] implies that l_{11} and l_{21} are coprime.

Consider the case $w_{02} \in \tau^-$. Then $\gamma_{02,1} \in \mathrm{rlv}(u)$ holds, Lemma 2.4.9 (ii) yields $l_{01} = 1$ and Remark 2.5.1 shows $w_{02}^1 = 1$. We conclude $\mu^1 = 2$ and thus obtain $l_{11} = l_{21} = 2$; a contradiction.

Now consider $w_{02} \in \tau^+$, which implies $\gamma_{01,02,11} \in \mathrm{rlv}(u)$. Since X is locally factorial, Remark 1.3.3 (ii) shows that w_{02}^2 and w_{11}^2 are coprime. Now we look at

$$\mu^2 \;=\; w_{02}^2 \;=\; l_{11}w_{11}^2 \;=\; l_{21}w_{21}^2.$$

We infer that l_{21} divides w_{02}^2 and w_{11}^2. This contradicts coprimeness of w_{02}^2 and w_{11}^2, because by irredundancy of P we have $l_{21} \geq 2$. □

Case (II). *We have $r = 3$, $m = 0$ and $2 = n_0 = n_1 \geq n_2 \geq n_3 \geq 1$. This leads to No. 13 in Theorems 2.1.1 and 2.1.2.*

Proof. We treat the constellations (a), (b) and (c) at once. First observe that for every $w_{i_1j_1}$ with $n_{i_1} = 2$, there is at least one $w_{i_2j_2}$ with $n_{i_2} = 2$ and $i_1 \neq i_2$ such that $\tau_X \subseteq Q(\gamma_{i_1j_1,i_2j_2})^\circ$ and thus $\gamma_{i_1j_1,i_2j_2} \in \mathrm{rlv}(u)$. Since $r = 3$, we conclude $l_{ij} = 1$ for all i with $n_i = 2$; see Lemma 2.4.9 (iv).

We can assume $w_{01}, w_{11} \in \tau^-$ and $w_{02}, w_{12} \in \tau^+$ as well as $w_{11} \in \mathrm{cone}(w_{01}, w_{12})$. Applying Remark 2.5.1 to $\gamma_{01,12}, \in \mathrm{rlv}(u)$, we obtain $w_{01} = (1,0)$ and $w_{12} = (0,1)$. Moreover $w_{11}^1, w_{11}^2 \geq 0$ holds and, because of $w_{11} \notin \tau^+$, we even have $w_{11}^1 > 0$. For the degree μ of g_0 and g_1 we note

$$\mu^1 \;=\; w_{02}^1 + 1 \;=\; w_{11}^1, \qquad\qquad \mu^2 \;=\; w_{02}^2 \;=\; w_{11}^2 + 1.$$

Thus, we can express w_{02} in terms of w_{11}. Remark 2.5.1 applied to $\gamma_{02,11} \in \mathrm{rlv}(u)$ gives $1 = \det(w_{11}, w_{02}) = w_{11}^1 + w_{11}^2$. We conclude $w_{11} = (1,0)$ and $w_{02} = (0,1)$. In particular, the degree of the relations g_0 and g_1 is $\mu = (1,1)$.

In constellations (b) and (c), we have $n_3 = 1$ and $\mu = (1,1)$. This implies $l_{31} = 1$, a contradiction to P being irredundant. Thus, constellations (b) and (c) do not occur.

We are left with constellation (a), that means that we have $n_0 = \ldots = n_3 = 2$. As seen before, $l_{ij} = 2$ for all i, j. Thus, the relations are

$$g_0 \;=\; T_{01}T_{02} + T_{11}T_{12} + T_{21}T_{22}, \qquad g_1 \;=\; \lambda T_{11}T_{12} + T_{21}T_{22} + T_{31}T_{32},$$

where $\lambda \in \mathbb{K}^* \setminus \{1\}$. In this situation, we may assume $w_{21}, w_{31} \in \tau^-$. Applying Remark 2.5.1 to the relevant faces $\gamma_{02,21}, \gamma_{02,31}$, we conclude $w_{21}^1 = w_{31}^1 = 1$. Since $\mu^1 = 1$ and $l_{ij} = 1$, we obtain $w_{22}^1 = w_{32}^1 = 0$. Thus, w_{22} and w_{32} lie in τ^+. Again Remark 2.5.1, this time applied to $\gamma_{01,22}, \gamma_{01,32} \in \mathrm{rlv}(u)$, yields $w_{22}^2 = w_{32}^2 = 1$. Since $\mu^2 = 1$ and $l_{ij} = 1$, we obtain $w_{21}^2 = w_{31}^2 = 0$. Hence we obtain the degree matrix

$$Q \;=\; \left[\begin{array}{cc|cc|cc|cc} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right].$$

The semiample cone is $\mathrm{SAmple}(X) = (\mathbb{Q}_{\geq 0})^2$ and the anticanonical divisor is $-\mathcal{K}_X = (2,2)$. In particular, X is a Fano variety. □

Proof of Theorems 2.1.1, 2.1.2 and 2.1.4. The preceding analysis of the cases of Proposition 2.4.1 shows that every smooth rational non-toric projective variety of Picard number two coming with a torus action of complexity one occurs in Theorem 2.1.1 and, among these, the Fano ones in Theorem 2.1.2 and the truly almost

Fano ones in Theorem 2.1.4. Comparing the defining data, one directly verifies that any two different listed varieties are not isomorphic to each other. Finally, using Remark 1.3.3 one explicitly checks that indeed all varieties listed in Theorem 2.1.1 are smooth. $\square$

CHAPTER 3

Smooth intrinsic quadrics of small Picard number

In this chapter we continue to work on classifications of smooth Mori dream spaces with small Picard number and investigate *intrinsic quadrics*, i.e. Mori dream spaces whose Cox rings admit K-homogeneous generators such that the associated ideal is generated by a single purely quadratic polynomial. In Picard number one, the situation is similar to the toric case: there is up to isomorphism exactly one smooth intrinsic quadric per dimension, see Proposition 3.2.1. In Picard number two, Theorem 3.2.8 gives a description of all smooth intrinsic quadrics, thereby generalizing a result of [**11**]. In Picard number three, we provide in Theorem 3.3.2 a description of all smooth full intrinsic quadrics, i.e. smooth intrinsic quadrics whose Cox rings do not admit free variables. Specializing to small dimensions, we present in Theorem 3.3.5 and Theorem 3.3.6 a complete list of all smooth intrinsic quadrics of Picard number three and dimension at most four. In both cases, we further describe the smooth (almost) Fano intrinsic quadrics.

While we present the main tools needed in our classifications for intrinsic quadrics in Section 3.1, Sections 3.2 and 3.3 contain the classification results. In Section 3.4, we take a closer look at the four-dimensional smooth Fano intrinsic quadrics and describe explicitly their elementary birational divisorial contractions and their elementary contractions of fiber type. The remaining part of Chapter three is devoted to the proof of our classification results for smooth intrinsic quadrics of Picard number three.

3.1. Basics on intrinsic quadrics

In the following we show that the defining quadratic polynomial of an intrinsic quadric can be assumed to have an especially nice form.

Definition 3.1.1. Let X be an irreducible normal projective variety with finitely generated divisor class group $K := \mathrm{Cl}(X)$ and finitely generated Cox ring $\mathcal{R}(X)$. If $\mathcal{R}(X)$ admits K-homogeneous generators such that the associated ideal of relations is generated by a single purely quadratic polynomial, then we call X an *intrinsic quadric.* A *standard intrinsic quadric* is an intrinsic quadric X with Cox ring

$$\mathcal{R}(X) \ = \ \mathbb{K}[T_1, \ldots, T_r, S_1, \ldots, S_t]/\langle g\rangle \, ,$$

where g is a K-homogeneous polynomial of the form $g = T_1T_2 + \ldots + T_{q-1}T_q + h$ with some $0 \leq q \leq r$, $r \geq 3$, and some polynomial h given by

$$h \ = \ \begin{cases} T_{q+1}^2 + \ldots + T_r^2 & \text{if } q < r \, , \\ 0 & \text{if } q = r \, , \end{cases}$$

where $\deg(T_{q+k}) \neq \deg(T_{q+l})$ holds for all $1 \leq k < l \leq r - q$. If X is a standard intrinsic quadric for which $t = 0$ holds, then we call X a *full intrinsic quadric.*

The following proposition shows that we can reduce the classification of intrinsic quadrics to the classification of standard intrinsic quadrics.

Proposition 3.1.2. *Let K be a finitely generated abelian group, consider a K-grading on the polynomial ring $\mathbb{K}[T_1, \ldots, T_s]$ such that the variables $T_1, \ldots, T_s$ and the following quadratic polynomial are K-homogeneous:*

$$g \;=\; \sum_{1 \le i \le j \le s} a_{ij} T_i T_j \;\in\; \mathbb{K}[T_1, \ldots, T_s]\,.$$

Then there is a linear automorphism $\psi\colon \operatorname{lin}(T_1, \ldots, T_s) \to \operatorname{lin}(T_1, \ldots, T_s)$ inducing an automorphism of K-graded algebras $\Psi\colon \mathbb{K}[T_1, \ldots, T_s] \to \mathbb{K}[T_1, \ldots, T_s]$ such that

$$\Psi(g) \;=\; T_1T_2 + \ldots + T_{q-1}T_q + h$$

holds for some $0 \le q \le r \le s$ and some polynomial h given by

$$h \;=\; \begin{cases} T_{q+1}^2 + \ldots + T_r^2 & \text{if } q < r\,, \\ 0 & \text{if } q = r\,, \end{cases}$$

where we have $\deg(T_{q+k}) \ne \deg(T_{q+l})$ for all $1 \le k < l \le r - q$.

Proof. Suitably renumbering the variables, we may assume that $T_1, \ldots, T_r$ are precisely the variables of g showing up in g. Denote by $w_1, \ldots, w_n \in K$ the degrees of $T_1, \ldots, T_r$, where $w_k \ne w_l$ holds for $k \ne l$. Moreover, set $\mu := \deg(g) \in K$. Suitable renumbering of variables yields

$$w_1 + w_2 = \ldots = w_m + w_{m+1} = \mu, \qquad 2w_{m+2} = \ldots = 2w_n = \mu$$

with a unique odd number $-1 \le m < n$. Some of the variables $T_1, \ldots, T_s$ may share the same degree and we have

$$V \;:=\; \operatorname{lin}(T_1, \ldots, T_s) \;=\; V_1 \oplus \ldots \oplus V_n \oplus V_0,$$

where V_k is the linear subspace generated by all T_i, $1 \le i \le r$, of degree w_k, and V_0 is the linear subspace generated by the variables $T_{r+1}, \ldots, T_s$. Suitably renumbering the T_i again, we obtain

$$T_1, \ldots, T_{d_1} \in V_1, \quad \ldots, \quad T_{d_{n-1}+1}, \ldots, T_{d_n} \in V_n, \quad T_{d_n+1}, \ldots, T_s \in V_0.$$

The idea is to build up ψ stepwise from appropriate endomorphisms $V \to V$. First, consider variables $T_i \in V_1$ and $T_j \in V_2$ with $\alpha_{ij} \ne 0$. Define a linear automorphism

$$\psi_{ij}\colon V \;\to\; V, \qquad T_j \;\mapsto\; a_{ij}^{-1} T_j - a_{ij}^{-1} \sum_{k \ne j} a_{ik} T_k, \quad T_l \;\mapsto\; T_l \text{ for } l \ne j.$$

Then ψ_{ij} respects the direct sum decomposition of V and restricts to the identity on all components different from V_2. Moreover, ψ_{ij} extends to an automorphism Ψ_{ij} of the K-graded algebra $\mathbb{K}[T_1, \ldots, T_s]$ and we have

$$\Psi_{ij}(g) \;=\; \left(T_i + a_{ij}^{-1} \sum_{k \ne i} a_{kj} T_k \right) T_j + \sum_{k \ne i, l \ne j} \tilde{a}_{kl} T_k T_l$$

with some $\tilde{a}_{kl} \in \mathbb{K}$. Now define a linear automorphism

$$\psi_{ji}\colon V \;\to\; V, \qquad T_i \;\mapsto\; T_i - a_{ij}^{-1} \sum_{k \ne i} a_{kj} T_k, \quad T_l \;\mapsto\; T_l \text{ for } l \ne i.$$

Similarly as before, ψ_{ji} respects the direct sum decomposition of V and restricts to the identity on all components different from V_1. Again, ψ_{ji} extends to an automorphism Ψ_{ji} of the K-graded algebra $\mathbb{K}[T_1, \ldots, T_s]$. This time we have

$$\Psi_{ji}(\Psi_{ij}(g)) \;=\; T_i T_j + \sum_{k \ne i,\ l \ne j} \tilde{a}_{kl} T_k T_l\,.$$

Thus, a suitable composition of the automorphisms $\Psi_{ji} \circ \Psi_{ij}$ turns g into the desired form with respect to the variables from V_1 and V_2. Proceeding similarly, we can settle all other pairs V_l and V_{l+1} for $l = 3, 5, \ldots, m$.

On each subspace V_k for $k > m+1$, the variables all have the same K-degree and, if a variable of a given monomial of g belongs to V_k, then all variables of this monomial belong to V_k. Thus, we may treat the part q_k of q built from variables of V_k separately. The usual diagonalization procedure for the Gram matrix of q_k leads to a presentation of q_k as a sum of squares. If the number of these squares is even, then we turn the whole q_k into a sum of terms T_iT_j with $i \neq j$. Otherwise, we turn q_k into a sum of T_iT_j with $i \neq j$ plus one single square. □

If X is an intrinsic quadric, then we can apply Proposition 3.1.2 to see that there is an automorphism of K-graded algebras mapping $\mathcal{R}(X)$ to the Cox ring of a standard intrinsic quadric. Thus the notion of standard intrinsic quadrics comprises the case of a general intrinsic quadric and we obtain the following:

Corollary 3.1.3. *Every intrinsic quadric is isomorphic to a standard intrinsic quadric.*

Remark 3.1.4. Assume that X is a standard intrinsic quadric. This means that its Cox ring is given as $\mathcal{R}(X) = \mathbb{K}[T_1, \ldots, T_r, S_1, \ldots, S_t]/\langle g \rangle$, where $g = T_1T_2 + \ldots + T_{q-1}T_q + h$ holds for some $0 \leq q \leq r$ and some polynomial h given by

$$h \;=\; \begin{cases} T_{q+1}^2 + \ldots + T_r^2 & \text{if } q < r\,, \\ 0 & \text{if } q = r\,, \end{cases}$$

where $\deg(T_{q+k}) \neq \deg(T_{q+l})$ holds for all $1 \leq k < l \leq r-q$. According to [**3**, Prop. 3.3.3.2], the anticanonical class of X is given by

$$-\mathcal{K}_X \;=\; (q/2-1)\deg(g) + \sum_{i=q+1}^{r} \deg(T_i) + \sum_{j=1}^{t} \deg(S_j) \;\in\; K.$$

Recall that we denote by $Q\colon \mathbb{Z}^{r+t} \to K = \mathrm{Cl}(X)$ the map defined through $e_i \mapsto w_i$, $e_{r+j} \mapsto u_j$, where e_i, e_{r+j} are the canonical base vectors of $E = \mathbb{Z}^{r+t}$ and where $w_i := \deg(T_i)$ and $u_j := \deg(S_j)$ denote the degrees of the generators of $\mathcal{R}(X)$. Furthermore, we set $\gamma := \mathbb{Q}^{r+t}_{\geq 0}$.

Lemma 3.1.5. *Assume that X is a standard intrinsic quadric with Cox ring given as in Remark 3.1.4. Let $\gamma_0 \preceq \mathbb{Q}^{r+t}_{\geq 0}$ be a face of the positive orthant. Then the piece $\overline{X}(\gamma_0)$ is singular if and only if $e_i \notin \gamma_0$ holds for all $i = 1, \ldots, r$.*

Proof. Let $z \in \overline{X}(\gamma_0)$. The claim follows since the gradient of g evaluated in z vanishes if and only if $e_i \notin \gamma_0$ holds for all $i = 1, \ldots, r$. □

Lemma 3.1.6. *Assume that X is a standard intrinsic quadric with Cox ring given as in Remark 3.1.4. Then X is smooth if and only if all elements $\gamma_0 \in \mathrm{cov}(u)$ fulfill the following two conditions:*

(i) *There is $1 \leq i \leq r$ such that $e_i \in \gamma_0$ holds.*
(ii) *Q maps $\mathrm{lin}(\gamma_0) \cap E$ onto K.*

Proof. According to Lemma 3.1.5, the first item is equivalent to X being quasi-smooth. Thus, Remark 1.3.3 completes the proof. □

Proposition 3.1.7. *Let X be a full intrinsic quadric. If X is Fano, then its Picard number is bounded by $\rho(X) \leq 3$. If $\rho(X) = 3$ holds and X is a full intrinsic Fano quadric, then X is $\mathbb{Q}$-factorial.*

Proof. The Cox ring of a full intrinsic quadric X is given as $\mathbb{K}[T_1, \ldots, T_r]/\langle g \rangle$, where $g = T_1T_2 + \ldots + T_{q-1}T_q + h$ holds for some $0 \leq q \leq r$ and some polynomial h given by

$$h \;=\; \begin{cases} T_{q+1}^2 + \ldots + T_r^2 & \text{if } q < r\,, \\ 0 & \text{if } q = r\,. \end{cases}$$

We have $\mathfrak{F} = (T_1, \ldots, T_r)$ and Φ consists of all projected $\mathfrak{F}$-faces $Q(\gamma_0)$, where $\gamma_0 \preceq \mathbb{Q}^r_{\geq 0}$ holds with

$$-\mathcal{K}_X \otimes 1 \;=\; ((r/2-1)\deg(g)) \otimes 1 \;\in\; Q(\gamma_0)^\circ$$

and $\deg(g)$ denotes the degree of g. We first discuss the case $h \neq 0$. Here, we look at $\gamma_0 := \mathrm{cone}(e_1, e_2, e_r)$. This is an $\mathfrak{F}$-face and we have $Q(\gamma_0) \in \Phi$. Because of

$$Q(e_1 + e_2 - 2e_r) \;=\; \deg(g) - \deg(g) \;=\; 0,$$

the image $Q(\mathrm{lin}(\gamma_0))$ is of dimension at most two. According to Proposition 1.3.2, the Picard group of X satisfies

$$\mathrm{Pic}(X) \;\subseteq\; Q(\mathrm{lin}(\gamma_0) \cap E),$$

i.e. the Picard number of X is at most two. Now, let $h = 0$, i.e. we have $g = T_1T_2 + \ldots + T_{r-1}T_r$. Consider the cones $\tau_{ij} := \mathrm{cone}(e_i, e_{i+1}, e_j, e_{j+1})$, where i, j are odd with $1 \leq i < j \leq r-1$. The τ_{ij} are $\mathfrak{F}$-faces and $Q(\tau_{ij})$ is contained in Φ. Because of

$$Q(e_i + e_{i+1} - e_j - e_{j+1}) \;=\; \deg(g) - \deg(g) \;=\; 0,$$

the images $Q(\mathrm{lin}(\tau_i))$ are of dimension at most three. Again by Proposition 1.3.2, we have

$$\mathrm{Pic}(X) \;\subseteq\; \bigcap_{i,j} Q(\mathrm{lin}(\tau_{ij}) \cap E),$$

i.e. the Picard number of X is at most three.

It remains to show that X is $\mathbb{Q}$-factorial if $\varrho(X) = 3$. In this case the above considerations show that h equals zero. Moreover, since $\rho(X) = 3$ holds, Remark 1.3.3 shows that the dimension of $Q(\tau_{ij})$ is three for all odd i, j with $1 \leq i < j \leq r-1$ and we conclude that the cones $Q(\tau_{ij})$ generate all the same three-dimensional vector subspace $V \subseteq K_{\mathbb{Q}}$. Thus $\dim(K_{\mathbb{Q}}) = 3$ follows from

$$K_{\mathbb{Q}} \;=\; Q(\mathbb{Q}^r) \;=\; Q(\mathrm{lin}_{\mathbb{Q}}(\tau_{13}) + \ldots + \mathrm{lin}_{\mathbb{Q}}(\tau_{r-3,r-1})) \;=\; V.$$

$\square$

3.2. Classification results in Picard number at most two

In this section we present our description of smooth intrinsic quadrics of Picard number at most two, see Proposition 3.2.1 and Theorem 3.2.8. In Picard number one, we prove that there is only one smooth intrinsic quadric per dimension. We further show that all these varieties are Fano, whereas our description in Picard number two reveals smooth intrinsic quadrics being not Fano. We further give descriptions of all smooth intrinsic (almost) Fano intrinsic quadrics in Picard number two, see Theorems 3.2.10 and 3.2.11. As an application we prove in Proposition 3.2.14 Mukai's conjecture for the smooth Fano intrinsic quadric of Picard at most two.

Proposition 3.2.1. *Let X be a smooth intrinsic quadric of Picard number one. Then X is isomorphic to the variety defined by the Cox ring*

$$\mathbb{K}[T_1, \ldots, T_r]/\langle T_1T_2 + T_3T_4 + \ldots + T_{i-1}T_i + h\rangle,$$

where $i = r-2$, $h = T_{r-1}T_r$ or $i = r-1$, $h = T_r^2$ holds, and where the grading is given by $\deg(T_j) = 1 \in \mathbb{Z} = \mathrm{Cl}(X)$ for all $1 \leq j \leq r$. In particular, X is Fano.

Proof. Let X be a smooth intrinsic quadric of Picard number one. According to Corollary 3.1.3, we may assume that X is a standard intrinsic quadric, i.e. its

Cox ring is given as $\mathcal{R}(X) = \mathbb{K}[T_1, \ldots, T_r, S_1, \ldots, S_t]/\langle g \rangle$, where $g = T_1T_2 + \ldots + T_{q-1}T_q + h$ holds for some $0 \leq q \leq r$, $r \geq 3$, and some polynomial h given by

$$h \;=\; \begin{cases} T_{q+1}^2 + \ldots + T_r^2 & \text{if } q < r\,, \\ 0 & \text{if } q = r\,, \end{cases}$$

where $\deg(T_{q+k}) \neq \deg(T_{q+l})$ holds for all $1 \leq k < l \leq r - q$.

In a first step, we show that $\mathrm{Cl}(X) = \mathbb{Z}$ holds. If $q > 0$ holds, then γ_1 is a one-dimensional relevant face. Since X is locally factorial, Remark 1.3.3 shows that we have $\mathrm{Cl}(X) = Q(\mathrm{lin}(\gamma_1) \cap E)$. Thus, we obtain $\mathrm{Cl}(X) = \mathbb{Z}$. Now we consider the case $q = 0$. Since g is homogeneous, we have $w_1^0 = w_j^0$ for all $j = 1, \ldots, r$. Furthermore, the cone $\gamma_{ij} := \mathrm{cone}(e_i, e_j)$ is a relevant face for all $1 \leq i < j \leq r$, where $e_i, e_j \in \mathbb{Z}^{r+t}$ denote as usual the canonical base vectors. This yields $\mathrm{lin}_{\mathbb{Z}}(w_i, w_j) \geq \mathbb{Z} \oplus \mathrm{Cl}(X)^{\mathrm{tor}}$ by Remark 1.3.3. In particular, we have $\mathrm{lin}_{\mathbb{Z}}(w_i^0) = \mathrm{lin}_{\mathbb{Z}}(w_i^0, w_j^0) \geq \mathbb{Z}$ for all $1 \leq i < j \leq r$. We conclude that $w_1^0 = 1$ holds for all $i = 1, \ldots, r$. Multiplying $(w_1, \ldots, w_r)$ with an unimodular matrix from the left, we arrive at

$$(w_1, \ldots, w_r) \;=\; \begin{pmatrix} 1 & 1 & \ldots & 1 \\ p & w_2^{\mathrm{tor}} & \ldots & w_r^{\mathrm{tor}} \end{pmatrix},$$

where $p = 0_{\mathrm{Cl}(X)^{\mathrm{tor}}}$ holds. Since $\mathrm{lin}_{\mathbb{Z}}(w_1, w_2) = \mathrm{lin}_{\mathbb{Z}}(w_1, w_i)$ holds for all $2 \leq i \leq r$, we conclude $w_2^{\mathrm{tor}} = w_i^{\mathrm{tor}}$ for all $2 \leq i \leq r$. This means that Remark 1.3.3 applied to γ_{23} shows

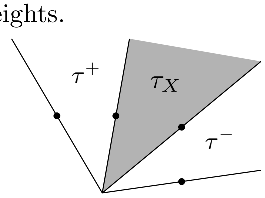
$$\mathrm{lin}_{\mathbb{Z}}(w_2) \;=\; \mathrm{lin}_{\mathbb{Z}}(w_2, w_3) \;\cong\; \mathbb{Z} \oplus \mathrm{Cl}(X)^{\mathrm{tor}}$$

holds, which implies that $\mathrm{Cl}(X)$ is torsion-free.

Since $\mathrm{Cl}(X)$ is torsion-free, g contains either zero or exactly one square. Remark 1.3.3 applied to γ_i, where T_i is not a square, shows that $w_i = 1$ holds for all i such that T_i is not a square. Homogeneity of g then yields $w_i = 1$ for all $1 \leq i \leq r$. Since X is smooth, Lemma 3.1.5 shows that $t = 0$ holds, i.e. there are no free variables.

Note that the anticanonical class of X is given by $-\mathcal{K}_X = r - 2$. Since g has at least three variables, $r - 2$ is contained in the relative interior of the semiample cone $\mathrm{SAmple}(X) = \mathbb{Q}_{\geq 0}$, which shows that X is Fano. □

From now on, this section treats the case of Picard number two. Thus, $\mathrm{Cl}_{\mathbb{Q}}(X)$ is of dimension two and the effective cone $\mathrm{Eff}(X)$ is uniquely decomposed into three convex sets $\mathrm{Eff}(X) = \tau^+ \cup \tau_X \cup \tau^-$ such that τ^+ and τ^- do not intersect the ample cone $\tau_X := \mathrm{Ample}(X)$ and $\tau^+ \cap \tau^-$ consists of the origin. The extremal rays of $\mathrm{Eff}(X)$ as well as the bounding rays of τ_X are generated by some of the weights w_i, u_j. Because of $\tau_X \subseteq \mathrm{Mov}(X)^\circ$, each of τ^+ and τ^- contains at least two (not necessarily different) weights.



Notation 3.2.2. Assume that X is a standard intrinsic quadric with Cox ring given as $\mathcal{R}(X) = \mathbb{K}[T_1, \ldots, T_r, S_1, \ldots, S_t]/\langle g \rangle$, where $g = T_1T_2 + \ldots + T_{q-1}T_q + h$ holds for some $0 \leq q \leq r$ and some polynomial h given by

$$h \;=\; \begin{cases} T_{q+1}^2 + \ldots + T_r^2 & \text{if } q < r\,, \\ 0 & \text{if } q = r\,, \end{cases}$$

where we have $\deg(T_{q+k}) \neq \deg(T_{q+l})$ for all $1 \leq k < l \leq r - q$. Consider the canonical base vectors $e_1, \ldots, e_{r+t} \in E = \mathbb{Z}^{r+t}$ and the positive orthant $\gamma := \mathbb{Q}_{\geq 0}^{r+t}$.

For indices $1 \le \ell_1 < \ell_2 < \ldots < \ell_s \le r+t$ we set

$$\gamma_{\ell_1\ell_2\ldots\ell_s} \ := \ \gamma_{\ell_1,\ell_2,\ldots,\ell_s} \ := \ \mathrm{cone}(e_{\ell_1},\ldots,e_{\ell_s}) \ \preceq \gamma\,,$$

where we use the notation in the middle instead of the one on the left-hand side in case further clarification is needed.

Remark 3.2.3. In the above notation, a face $\tau \preceq \gamma$ is an $\mathfrak{F}$-face of X if and only if one of the following criteria is fulfilled:

(i) There are odd indices $1 \le i < j \le q$ such that $\gamma_{i,i+1,j,j+1} \preceq \tau$ holds.
(ii) There is an odd index $1 \le i \le q$ and an index $q+1 \le j \le r$ such that $\gamma_{i,i+1,j} \preceq \tau$ holds.
(iii) There are indices $q+1 \le i < j \le r$ such that $\gamma_{ij} \preceq \tau$ holds.
(iv) For each odd $1 \le i \le q-1$ there is an index $i \le k_i \le i+1$ such that τ is a face of $\gamma_{k_1,\ldots,k_{q-1},r+1,\ldots,r+t}$.

Remark 3.2.4. Assume that X is a standard intrinsic quadric and assume that X is $\mathbb{Q}$-factorial. If $\gamma_{\ell_1\ell_2\ldots\ell_s}$ is a relevant face, then Remark 1.3.3 implies that the family $(w_{\ell_1},\ldots,w_{\ell_s})$ generates a full-dimensional cone in $\mathrm{Cl}(X)_{\mathbb{Q}}$. Thus $\mathfrak{F}$-faces $\gamma_0 \preceq \gamma$ for which $Q(\gamma_0)$ is not of the same dimension as $\mathrm{Cl}(X)_{\mathbb{Q}}$ are not a relevant faces. In particular, if u is an ample Weil divisor class and if $\gamma_0 \preceq \gamma$ is an $\mathfrak{F}$-face such that $u \in Q(\gamma_0)$ holds and such that all faces of γ_0 are also $\mathfrak{F}$-faces, then we obtain $u \in Q(\gamma_0)^\circ$.

Remark 3.2.5. Assume that X is a standard intrinsic quadric. If $\gamma_{\ell_1\ell_2\ldots\ell_s}$ is a relevant face and if X is locally factorial, then Remark 1.3.3 implies that the family $(w_{\ell_1},\ldots,w_{\ell_s})$ generates $K = \mathrm{Cl}(X)$ as an abelian group. In particular, if $s = \rho(X)$ holds, then K is torsion-free and we have

$$\pm 1 \ = \ \det(w_{\ell_1},\ldots,w_{\ell_{\rho(X)}})\,.$$

Since multiplying Q from the left with an unimodular matrix does not affect the isomorphism type of the underlying Mori dream space, we may then assume that $w_{\ell_1},\ldots,w_{\ell_{\rho(X)}}$ are the canonical base vectors of $K \cong \mathbb{Z}^{\rho(X)}$.

Proposition 3.2.6. *Let X be an intrinsic quadric of Picard number two. If X is locally factorial, then* $\mathrm{Pic}(X) = \mathbb{Z}^2$ *holds.*

Proof. Corollary 3.1.3 shows that we may assume that X is a standard intrinsic quadric, i.e. its Cox ring is given as $\mathcal{R}(X) = \mathbb{K}[T_1,\ldots,T_r,S_1,\ldots,S_t]/\langle g\rangle$, where $g = T_1T_2 + \ldots + T_{q-1}T_q + h$ holds for some $0 \le q \le r$, $r \ge 3$, and some polynomial h given by

$$h \ = \ \begin{cases} T_{q+1}^2 + \ldots + T_r^2 & \text{if } q < r\,,\\ 0 & \text{if } q = r\,,\end{cases}$$

where $\deg(T_{q+k}) \neq \deg(T_{q+l})$ holds for all $1 \le k < l \le r-q$. According to Remark 3.2.5 it is sufficient to show that there is a two-dimensional relevant face. Let $u \in \mathrm{Cl}(X)$ be an ample Weil divisor class. We distinguish the two following two cases:

(1) g consists of squares,
(2) after renumbering of variables we have $g = T_1T_2 + \ldots$.

Case (1): According to Carathéodory's theorem, there is an at most two-dimensional face τ of the positive orthant $\mathbb{Q}^{r+t}_{\ge 0}$ such that $u \in Q(\tau)^\circ$ holds. If τ is an $\mathfrak{F}$-face, then τ is a relevant face. Since X is $\mathbb{Q}$-factorial, τ then is two-dimensional and thus the proof is complete.

If τ is not an $\mathfrak{F}$-face, then, possibly after renumbering of variables, we have $\tau = \gamma_1$ or $\tau = \gamma_{1,r+1}$, where $u_1 = Q(e_{r+1})$ denotes the weight corresponding to the free variable S_1. We show that only the second choice for τ is possible: If we had $u \in Q(\gamma_1)^\circ$, then γ_{12} would be a relevant face, contradicting Remark 3.2.4. Thus we are in situation two, i.e. $\tau = \gamma_{1,r+1}$ holds. Note that we have $Q(\gamma_{i,j,r+1})^\circ = Q(\tau)^\circ$ for all $1 \le i < j \le r$, which shows that $\gamma_{i,j,r+1}$ is a relevant face for all $1 \le i < j \le r$. This yields $\mathrm{lin}_\mathbb{Z}(w_i, w_j, u_1) \ge \mathbb{Z}^2 \oplus \mathrm{Pic}(X)^{\mathrm{tor}}$ by Remark 3.2.5. Since g is homogeneous, we have $w_1^0 = w_i^0$ for all $i = 1, \ldots, r$. In particular, we obtain $\mathrm{lin}_\mathbb{Z}(w_i^0, u_1^0) \ge \mathbb{Z}^2$ for all $1 \le i < j \le r$. Multiplying Q with an unimodular matrix from the left, we arrive at

$$(w_1, \ldots, w_r \,|\, u_1) \ = \ \left(\begin{array}{cccc|c} 0 & 0 & \ldots & 0 & 1 \\ 1 & 1 & \ldots & 1 & 0 \\ p & w_2^{\mathrm{tor}} & \ldots & w_r^{\mathrm{tor}} & p \end{array}\right),$$

where $p = 0_{\mathrm{Pic}(X)^{\mathrm{tor}}}$ holds. Since $\mathrm{lin}_\mathbb{Z}(w_1, w_2, u_1) = \mathrm{lin}_\mathbb{Z}(w_1, w_i, u_1)$ holds for all $2 \le i \le r$, we conclude $w_2^{\mathrm{tor}} = w_i^{\mathrm{tor}}$ for all $2 \le i \le r$. This means that Remark 3.2.5 applied to $\gamma_{2,3,r+1}$ yields

$$\mathrm{lin}_\mathbb{Z}(w_2, u_1) \ = \ \mathrm{lin}_\mathbb{Z}(w_2, w_3, u_1) \ \cong \ \mathbb{Z}^2 \oplus \mathrm{Pic}(X)^{\mathrm{tor}},$$

which implies that $\mathrm{Pic}(X)$ is torsion-free.

Case (2): Here we have

$$\mathrm{Eff}(X) \ = \ Q(\sigma) \qquad \text{with} \qquad \sigma \ := \ \mathrm{cone}\big(e_i;\ T_i^2 \text{ is not a square}\big).$$

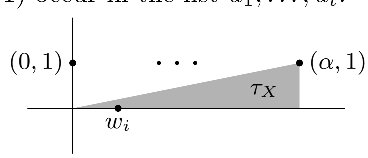
Carathéodory's theorem shows that there is an at most two-dimensional face τ of σ such that $u \in Q(\tau)^\circ$ holds. If τ is an $\mathfrak{F}$-face, then Remark 3.2.4 implies that τ is a two-dimensional relevant face, which completes the proof in this situation. If τ is not an $\mathfrak{F}$-face, then, possibly after renumbering of variables, we have $\tau = \gamma_{12}$, where $g = T_1T_2 + \ldots$ holds. We may assume that w_1 is contained in τ^+ and w_2 in τ^-. Since $u \in \mathrm{Mov}(X)^\circ$ holds, there is a further weight $w_+ \in \tau^+$. If T_+^2 is not a square of g, then $\gamma_{2,+}$ is a two-dimensional relevant face. If T_+^2 is a square of g, then we consider a further weight $w_- \in \tau^-$. Since $\deg(g)$ lies in τ^+, w_- does not belong to a square. Thus, $\gamma_{1,-}$ is a two-dimensional relevant face. $\square$

Construction 3.2.7. Fix two integers $r \in \mathbb{Z}_{\ge 5}$ and $t \in \mathbb{Z}_{\ge 0}$. Consider the $\mathbb{K}$-algebra $R := \mathbb{K}[T_1, \ldots, T_r, S_1, \ldots, S_t]/\langle g \rangle$, where

$$g \ := \ \begin{cases} T_1T_2 + \ldots + T_{r-1}T_r & \text{if } r \text{ is even}, \\ T_1T_2 + \ldots + T_{r-2}T_{r-1} + T_r^2 & \text{if } r \text{ is odd}, \end{cases}$$

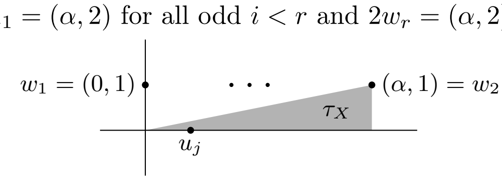
holds for some integers $r \in \mathbb{Z}_{\ge 5}$ and $t \in \mathbb{Z}_{\ge 0}$. Furthermore, a $\mathbb{Z}^2$-grading of R is obtained by choosing weights $w_i = \deg(T_i)$ and $u_j = \deg(S_j)$ according to one of the following settings.

Setting 1: Fix $\alpha \in \mathbb{Z}_{\ge 0}$. The weights u_j are taken from $(a, 1)$, where $0 \le a \le \alpha$ holds and we have $w_i = (1, 0)$ for all $1 \le i \le r$. Furthermore, we have $t \ge 2$ and the vectors $(\alpha, 1)$ and $(0, 1)$ occur in the list $u_1, \ldots, u_t$.



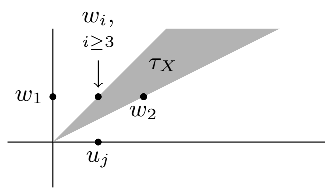
Setting 2: Fix $\alpha \in \mathbb{Z}_{\ge 0}$. The weights w_i are taken from $(a, 1)$, where $0 \le a \le \alpha$ holds and we have $u_j = (1, 0)$ for all $1 \le j \le t$. Furthermore, we have $t \ge 2$ and the weights satisfy

(i) $w_1 = (0, 1)$ and $w_2 = (\alpha, 1)$,
(ii) $w_i + w_{i+1} = (\alpha, 2)$ for all odd $i < r$ and $2w_r = (\alpha, 2)$ if r is odd.



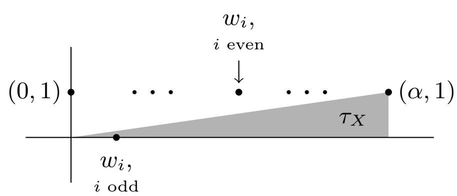
Setting 3: The weights w_i and u_j satisfy

(i) $w_1 = (0,1)$ and $w_2 = (2,1)$,
(ii) $w_i = (1,1)$ for all $3 \leq i \leq r$,
(iii) $u_j = (1,0)$ for all $1 \leq j \leq t$ and we have $t \geq 1$.



Setting 4: Here, $r \in \mathbb{Z}_{\geq 6}$ is even. The weights u_j are taken from $(a, 1)$, where $0 \leq a \leq \alpha$ holds with some $\alpha \in \mathbb{Z}_{\geq 0}$. We have $w_1 = (1,0)$ and $w_2 = (w_2^1, 1)$ for some $0 \leq w_2^1 \leq \alpha$. Furthermore the weights satisfy

(i) $w_i = w_1$ for all odd $1 \leq i \leq r-1$ and $w_i = w_2$ for all even $2 \leq i \leq r$,
(ii) the vectors $(\alpha, 1)$ and $(0,1)$ occur in the list $w_1, \ldots, w_r, u_1, \ldots, u_t$.



In all settings, g is $\mathbb{Z}^2$-homogeneous and R is the Cox ring of a smooth intrinsic quadric X with ample cone $\tau_X \subseteq \mathbb{Q}^2$ as indicated in the above figures.

Theorem 3.2.8 provides a classification of all smooth intrinsic quadrics, thereby generalizing a result of **[11]** that described the case of full intrinsic quadrics; i.e. precisely the examples with $\alpha = m = 0$ of Setting 4 of the above construction. Moreover, the cases $n = 5$ and $n = 6$ in Settings 1 to 4 are the ones allowing a torus action of complexity one and thus are exactly the overlap with the description presented in Chapter two: Setting 1 corresponds to Nos. 8 and 11, Setting 2 to Nos. 9 and 12, Setting 3 to Nos. 7 and 10 and Setting 4 to No. 4.

Theorem 3.2.8. *Let X be a smooth intrinsic quadric of Picard number two. Then X is isomorphic to an intrinsic quadric arising from Construction 3.2.7.*

Before presenting a proof of the above theorem, we first discuss some applications including the description of the smooth Fano and smooth almost Fano intrinsic quadrics of Picard number two.

Remark 3.2.9. All smooth intrinsic quadrics of Picard number two admit elementary contractions some of which we describe in this Remark.

Setting 1	There is a contraction of fiber type $\varphi\colon X \to V_{\mathbb{P}_{r-1}}(g)$ with fibers isomorphic to $\mathbb{P}_{t-1}$.
Setting 2	There is a contraction of fiber type $\varphi\colon X \to \mathbb{P}_{t-1}$ with fibers isomorphic to $V_{\mathbb{P}_{r-1}}(g)$.
Setting 3	There is a birational divisorial contraction $\varphi\colon X \to \mathbb{P}_{r+t-3}$ with center isomorphic to $V_{\mathbb{P}_{r-3}}(g - T_1T_2)$.
Setting 4	There is a contraction of fiber type $\varphi\colon X \to \mathbb{P}_{r/2-1}$ with fibers isomorphic to $\mathbb{P}_{r/2+t-2}$.

Theorem 3.2.10. *Let X be a smooth intrinsic quadric of Picard number two. Then X is Fano if and only if X is isomorphic to one of the following varieties arising from Construction 3.2.7:*

(i) *X arises from Setting 1 and we have $t\alpha < r - 2 + \sum_{j=1}^{t} u_j^1$.*
(ii) *X arises from Setting 2 and we have $(r/2-1)\alpha < t$.*
(iii) *X arises from Setting 3 and $r - 2 > t$ holds.*
(iv) *X arises from Setting 4, $\alpha t < (r/2-1) + \sum_{j=1}^{t} u_j^1$ and $w_2 = (\alpha, 1)$ hold.*

Theorem 3.2.11. *Let X be a smooth intrinsic quadric of Picard number two. Then X is truly almost Fano if and only if X isomorphic to one of the following varieties arising from Construction 3.2.7:*

(i) *X arises from Setting 1 and we have $t\alpha = r - 2 + \sum_{j=1}^{t} u_j^1$.*
(ii) *X arises from Setting 2 and we have $(r/2-1)\alpha = t$.*
(iii) *X arises from Setting 3 and $r - 2 = t$ holds.*
(iv) *X arises from Setting 4, $\alpha t = (r/2-1) + \sum_{j=1}^{t} u_j^1$ and $w_2 = (\alpha, 1)$ hold.*
(v) *X arises from Setting 4, $w_2 = (0,1)$ and $u_j = (1,1)$ hold for all $1 \le j \le t$.*

Proof of Theorems 3.2.10 and 3.2.11. All smooth intrinsic quadrics of Picard number two as well as their semiample cones are listed in Construction 3.2.7. Furthermore, recall that the anticanonical class of X is given by

$$-\mathcal{K}_X = (r/2-1)\deg(g) + \sum_{j=1}^{t} \deg(u_j)\,.$$

In order to select the Fano and the truly almost Fano varieties among the varieties in Construction 3.2.7, it is enough to compute the anticanonical class of X via the above formula and to check in which cases $\mathcal{K}_X \in \mathrm{SAmple}(X)^\circ$ and $\mathcal{K}_X \in \mathrm{SAmple}(X) \setminus \mathrm{SAmple}(X)^\circ$ holds.

In Setting 1, $w_i = (1,0)$ holds for all $1 \le i \le r$. We have $-\mathcal{K}_X = (r/2-1)(2,0) + \sum_{j=1}^{t}(u_j^1, 1)$ and $\mathrm{SAmple}(X) = \mathrm{cone}((1,0),(\alpha,1))$. This shows that X is Fano if and only if $t\alpha < r - 2 + \sum_{j=1}^{t} u_j^1$ holds and truly almost Fano if and only if $t\alpha = r - 2 + \sum_{j=1}^{t} u_j^1$ holds.

In Setting 2, $u_j = (1,0)$ holds for all $1 \le j \le t$. We have $-\mathcal{K}_X = (r/2-1)(\alpha,2) + t(1,0)$ and $\mathrm{SAmple}(X) = \mathrm{cone}((1,0),(\alpha,1))$. This shows that X is Fano if and only if $(r/2-1)\alpha < t$ holds and truly almost Fano if and only if $(r/2-1)\alpha = t$ holds.

In Setting 3, we have $-\mathcal{K}_X = (r-2+t, r-2)$ and the semiample cone of X is given by $\mathrm{SAmple}(X) = \mathrm{cone}((1,1),(2,1))$. Note that $-\mathcal{K}_X \in \mathrm{cone}((1,1),(1,0))^\circ$ holds. Thus, X is Fano if and only if $r - 2 > t$ holds and truly almost Fano if and only if $r - 2 = t$ holds.

In Setting 4, we have $-\mathcal{K}_X = (r/2-1)\deg(g) + \sum_{j=1}^{t} u_j$, where the degree of g is given by $\deg(g) = (w_2^1 + 1, 1)$ and the semiample cone of X by $\mathrm{SAmple}(X) = \mathrm{cone}((1,0),(\alpha,1))$. Note that u_j is not contained in the relative interior of the semiample cone of X. Furthermore, $\deg(g) \in \mathrm{Ample}(X)$ holds if and only if $w_2^1 = \alpha$ holds. Thus X is Fano if and only if $w_2^1 = \alpha$ and

$$(r/2-1)(\alpha+1,1) + (\sum_{j=1}^{t} u_j^1, t) \ \in \ \mathrm{SAmple}(X)^\circ$$

holds, where the latter is equivalent to $\alpha t < (r/2-1) + \sum_{j=1}^{t} u_j^1$. There are two possibilities for X being truly almost Fano in Setting 4: The first is that $w_2^1 = \alpha$ and $\alpha t = (r/2-1) + \sum_{j=1}^{t} u_j^1$ hold and the second that $w_2 = (0,1)$ as well as $u_j = (1,1)$ hold for all $1 \le j \le t$. $\square$

Remark 3.2.12. Recall that according to Corollary 2.1.3, any smooth rational non-toric Fano variety of Picard number two admitting a torus action of complexity one arises via iterated duplication of a free weight from a smooth rational projective (not necessarily Fano) variety with a torus action of complexity one, Picard number two and dimension at most seven. In Remark 2.2.7 we showed that there is no analogous statement for smooth toric Fano varieties of Picard number two. The same holds for smooth Fano intrinsic quadrics of Picard number two: Setting 4 of Construction 3.2.7 gives rise to the following series of smooth Fano intrinsic quadrics that cannot be constructed via duplication of free weights. For any $n \in \mathbb{Z}_{\geq 3}$ we obtain a full smooth intrinsic Fano quadric X_n of dimension $2n-3$ with Cox ring

$$\mathcal{R}(X_n) \ = \ \mathbb{K}[T_1, \ldots, T_{2n}]/\langle T_1T_2 + \ldots + T_{2n-1}T_{2n} \rangle \,,$$

semiample cone $\mathbb{Q}^2_{\geq 0}$ and generator degrees $\deg(T_i) = (1,0)$ for odd i and $\deg(T_i) = (0,1)$ for even i. Note that the anticanonical class is given as $\mathcal{K}_{X_n} = (n-1, n-1)$ which shows that X_n is Fano.

In the below corollary, the first few coefficients of the Hilbert series $H(t)$ were computed using the function `GRgradedcompdim` of `MDSpackage` [**38**].

Corollary 3.2.13. *Every smooth Fano intrinsic quadric of Picard number two and dimension at most four is isomorphic to one of the following varieties X, specified by their Cox ring $\mathcal{R}(X)$ and their anticanonical class $-\mathcal{K}_X$, where the grading is fixed by the matrix $Q = [w_1, \ldots, w_s]$, $s = dim(X)+3$, of generator degrees* $\deg(T_i) = w_i \in \mathrm{Cl}(X)$. *As additional data, we list the Fano index $q(X)$ and the first few terms of the Hilbert series $H(t)$.*

Setting	$\mathcal{R}(X)$	$Q = [w_1, \ldots, w_s]$	$-\mathcal{K}_X$	$q(X)$	$H(t)$
3	$\frac{\mathbb{K}[T_1,\ldots,T_6]}{\langle T_1T_2+T_3T_4+T_5^2 \rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|c} 0 & 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{array}\right]$	$(4,3)$	1	$1 + 26t + 120t^2$ $+329t^3 + 699t^4 + \ldots$
4	$\frac{\mathbb{K}[T_1,\ldots,T_6]}{\langle T_1T_2+T_3T_4+T_5T_6 \rangle}$	$\left[\begin{array}{cc\|cc\|cc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right]$	$(2,2)$	2	$1 + 27t + 125t^2$ $+343t^3 + 729t^4 + \ldots$
1	$\frac{\mathbb{K}[T_1,\ldots,T_7]}{\langle T_1T_2+T_3T_4+T_5^2 \rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|cc} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$	$(3,2)$	1	$1 + 90t + 700t^2$ $+2695t^3 + \ldots$
1	$\frac{\mathbb{K}[T_1,\ldots,T_7]}{\langle T_1T_2+T_3T_4+T_5^2 \rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|cc} 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$	$(4,2)$	2	$1 + 99t + 775t^2 + \ldots$
1	$\frac{\mathbb{K}[T_1,\ldots,T_7]}{\langle T_1T_2+T_3T_4+T_5^2 \rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|cc} 1 & 1 & 1 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$	$(5,2)$	1	$1 + 126t + 1000t^2 + \ldots$
2	$\frac{\mathbb{K}[T_1,\ldots,T_7]}{\langle T_1T_2+T_3T_4+T_5^2 \rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|cc} 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{array}\right]$	$(2,3)$	1	$1 + 90t + 700t^2 + \ldots$
3	$\frac{\mathbb{K}[T_1,\ldots,T_7]}{\langle T_1T_2+T_3T_4+T_5^2 \rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|cc} 0 & 2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{array}\right]$	$(5,3)$	1	$1 + 90t + 701t^2 + \ldots$
3	$\frac{\mathbb{K}[T_1,\ldots,T_7]}{\langle T_1T_2+T_3T_4+T_5T_6 \rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|c} 0 & 2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{array}\right]$	$(5,4)$	1	$1 + 90t + 699t^2 + \ldots$
4	$\frac{\mathbb{K}[T_1,\ldots,T_7]}{\langle T_1T_2+T_3T_4+T_5T_6 \rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|c} 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \end{array}\right]$	$(2,3)$	1	$1 + 90t + 700t^2$ $+2695t^3 + \ldots$
4	$\frac{\mathbb{K}[T_1,\ldots,T_7]}{\langle T_1T_2+T_3T_4+T_5T_6 \rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|c} 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \end{array}\right]$	$(4,3)$	1	$1 + 90t + 700t^2 + \ldots$

In particular, we see that there are ten smooth Fano intrinsic quadrics of Picard number two and dimension at most four and that all but two of them have Fano index 1.

Proposition 3.2.14. *Let X be a smooth Fano intrinsic quadric of Picard number at most two. Then X fulfills Mukai's conjecture, Conjecture 2.3.5, i.e. we have*

$$\rho(X)(q(X)-1) \ \leq \ \dim(X) \,.$$

Proof. In Proposition 3.2.1 we showed that in Picard number one, there is only one smooth intrinsic quadric X per dimension with $-\mathcal{K}_X = r-2 = \dim(X)$. We obtain the Fano index $q(X) = r-2$, i.e. X fulfills Mukai's conjecture. Now let X

be a smooth intrinsic quadric of Picard number two. By going through the settings of Theorem 3.2.10, we show that X fulfills Mukai's conjecture. Note that in all settings, we have $\dim(X) = r + t - 3$.

Assume that X arises from Setting 1. The Fano condition is $t\alpha < r - 2 + \sum_{j=1}^{t} u_j^1$. We distinguish the cases $\alpha = 0$ and $\alpha > 0$.

If $\alpha = 0$ holds, then we have $-\mathcal{K}_X = (r-2, t)$ and $q(X) = gcd(r-2, t)$. Note that this gives

$$2(q(X)-1) \;\leq\; 2\min(r-2,t) - 2 \;\leq\; (r-2+t) - 2 \;<\; \dim(X)\,.$$

Now consider the case $\alpha > 0$. Here we have $-\mathcal{K}_X = (r-2+\sum_{j=1}^{t} u_j^1,\ t)$ and $q(X) = gcd(r-2+\sum_{j=1}^{t} u_j^1, t)$. If $q(X) < t$ holds, then we obtain

$$2(q(X)-1) \;\leq\; 2(t/2-1) \;=\; t-2\,.$$

Since $r \geq 5$ holds, this yields $t - 2 \leq r + t - 7 < \dim(X)$, i.e. X fulfills Mukai's conjecture. If $q(X) = t$ holds, then t divides $r - 2 + \sum_{j=1}^{t} u_j^1$. The Fano condition shows that we have $r - 2 + \sum_{j=1}^{t} u_j^1 = \beta t$ for some $\beta \in \mathbb{Z}_{>\alpha}$. In particular, we have $r - 2 + \sum_{j=1}^{t} u_j^1 \geq (\alpha+1)t$. Thus we obtain

$$\begin{aligned} 2(q(X)-1) &= 2t-2 \\ &= (\alpha+1)t - (\alpha+1-2)t - 2 \\ &\leq \Big(r-2+\sum_{j=1}^{t} u_j^1\Big) - (\alpha+1-2)t - 2 \\ &< r-2+t-2 \\ &= \dim(X) - 1\,, \end{aligned}$$

where the last inequality follows since $\alpha > 0$ implies $\sum_{j=1}^{t} u_j^1 < \alpha t$. This completes the proof in Setting 1.

Now consider X arising from Setting 2. The Fano condition is $(r/2-1)\alpha < t$ and we have $-\mathcal{K}_X = ((r/2-1)\alpha + t,\ r-2)$. First we consider the case $q(X) < r-2$. Here we have

$$2(q(X)-1) \;\leq\; 2((r-2)/2-1) \;=\; r-4 \;\leq\; r+t-6\,,$$

where the last inequality follows since $t \geq 2$ holds. It remains to consider the case $q(X) = r-2$. If $\alpha = 0$ holds, then we obtain $-\mathcal{K}_X = (t, r-2)$ and thus $r - 2 \leq t$. We conclude

$$2(q(X)-1) \;=\; 2((r-2)-1) \;=\; (r-2)+(r-4) \;\leq\; t+r-4 \;<\; \dim(X)\,.$$

Now let $\alpha = 1$. Since $q(X)$ is the greatest common divisor of the two coordinates of $-\mathcal{K}_X = ((r-2)/2 + t, r-2)$, we obtain

$$t \;=\; \frac{2k+1}{2}(r-2)$$

with some $k \in \mathbb{Z}$. Because of $\alpha > 0$, the Fano condition shows that $t > (r-2)/2$ and thus $k \geq 1$ holds. Hence we obtain

$$\begin{aligned} 2(q(X)-1) &= 2(r-2-1) \\ &< 3(r-2)/2 + (r-2)/2 \\ &\leq t + r/2 - 1\,. \end{aligned}$$

The last expression is strictly smaller than $\dim(X)$ since $r > 4$ and thus $r/2 - 1 < r - 3$ holds. In Setting 2, it remains to consider the case $\alpha \geq 2$, $q(X) = r-2$. Here, the Fano condition ensures $r - 2 < 2t/\alpha$. Thus, we obtain

$$2(q(X)-1) \;=\; (r-2)+(r-4) \;<\; 2t/\alpha + (r-4) \;\leq\; t+r-4\,,$$

where the last inequality is true because of $\alpha \geq 2$.

In Setting 3, we have $-\mathcal{K}_X = (r-2+t,\ r-2)$ and the Fano condition is $r-2 > t$. Note that we have $r-2 < r-2+t < 2(r-2)$, which gives $q(X) = \gcd(r-2+t, r-2) < r-2$. We obtain

$$2(q(X)-1) \ \leq\ 2((r-2)/2-1) \ =\ r-4 \ <\ \dim(X)\,,$$

where the last inequality follows since t is at least one.

In Setting 4, we have $-\mathcal{K}_X = ((r/2-1)(w_2^1+1) + \sum_{j=1}^t u_j^1,\ r/2-1+t)$ and the Fano condition is $\alpha t < (r/2-1) + \sum_{j=1}^t u_j^1$, $w_2 = (\alpha, 1)$. If $\alpha = 0$ holds, then we obtain $q(X) \leq r/2-1$ and hence

$$2(q(X)-1) \ \leq\ r-4 \ \leq\ r+t-4 \ <\ \dim(X)\,.$$

If $\alpha > 0$ holds, then we distinguish the cases $q(X) < r/2-1+t$ and $q(X) = r/2-1+t$. In the first case, i.e. if $q(X)$ is strictly smaller than $r/2-1+t$, we obtain $q(X) \leq 1/2(r/2-1+t)$ and thus

$$2(q(X)-1) \ \leq\ r/2-1+t-2 \ <\ \dim(X)\,,$$

where the last inequality follows because of $r > 0$.

It remains to consider the case $\alpha > 0$, $q(X) = r/2-1+t$ in Setting 4. Note that $q(X)$ divides $(-\mathcal{K}_X)_1$, which means that we have $\beta(r/2-1+t) = (-\mathcal{K}_X)_1$ for some $\beta \in \mathbb{Z}$. The Fano condition shows $\beta \geq \alpha+1$. We conclude

$$\begin{array}{rcl} (\alpha+1)q(X) & \leq & \beta(r/2-1+t) \\ & = & (r/2-1)(w_2^1+1) + \displaystyle\sum_{j=1}^t u_j^1 \\ & < & (r/2-1)(w_2^1+1) + \alpha t\,, \end{array}$$

where the last inequality follows because $w_2 = (\alpha, 1)$ and $\alpha > 0$ show that there is some $1 \leq j \leq t$ with $u_j^1 = 0$. With this, we obtain

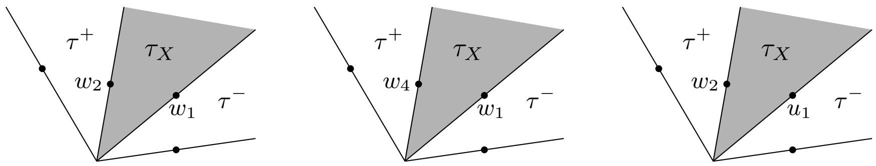
$$\begin{array}{rcl} 2(q(X)-1) & = & \big((\alpha+1) - (\alpha+1-2)\big)q(X) - 2 \\ & < & \big((r/2-1)(\alpha+1) + \alpha t\big) - (\alpha-1)q(X) - 2 \\ & = & r+t-4 \\ & < & \dim(X)\,, \end{array}$$

which completes the proof. $\square$

Proof of Theorem 3.2.8. Let X be a smooth intrinsic quadric of Picard number two. Proposition 3.2.6 guarantees that $\mathrm{Cl}(X)$ is torsion-free. Taking into account Corollary 3.1.3, we thus may assume that the defining relation of the Cox ring $R := \mathbb{K}[T_1, \ldots, T_r, S_1, \ldots, S_t]/\langle g \rangle$ is given by

$$g \ :=\ \begin{cases} T_1T_2 + \ldots + T_{r-1}T_r & \text{if } r \text{ is even,} \\ T_1T_2 + \ldots + T_{r-2}T_{r-1} + T_r^2 & \text{if } r \text{ is odd.} \end{cases}$$

Note that we have $r \geq 5$, because $\mathrm{Cl}(X)$ is torsion-free and thus $\mathcal{R}(X)$ must be a unique factorization domain. Since X is $\mathbb{Q}$-factorial, the ample cone $\tau_X \subseteq \mathrm{Cl}_{\mathbb{Q}}(X)$ is of dimension two. We work with the convex sets $\mathrm{Eff}(X) = \tau^+ \cup \tau_X \cup \tau^-$ as explained above. Lemma 3.1.5 shows that either all u_j are contained in τ^+ or all u_j are contained in τ^-. After suitably renumbering the variables T_i and S_j, we are left with the following cases.



(i) (ii) (iii)

We now go through the cases using Notation 3.2.2 for the relevant faces of X and denote by $\mu = (\mu_1, \mu_2) := \deg(g)$ the degree of g.

Case (i): We have $\tau_X = \mathrm{cone}(w_1, w_2)$, $w_1 \in \tau^-$ and $w_2 \in \tau^+$. Note that μ is contained in τ_X. We may thus assume that $w_3 \in \tau^-$ and $w_4 \in \tau^+$ hold. Applying Remark 3.2.5 to γ_{14}, we arrive at $w_1 = (1,0)$ and $w_4 = (0,1)$. Since g is homogeneous of degree μ, we obtain $w_2 = (\mu_1 - 1, \mu_2)$ and $w_3 = (\mu_1, \mu_2 - 1)$. Like w_1, w_4 also w_3, w_2 form a $\mathbb{Z}$-basis for $\mathrm{Cl}(X)$, being positively oriented, because $\mathrm{Eff}(X)$ is pointed and we have $w_2 \in \tau^+$ and $w_3 \in \tau^-$. This implies

$$1 \ = \ \det(w_3, w_2) \ = \ \mu_1 + \mu_2 - 1\,.$$

From $\mu \in \tau_X^\circ \subseteq \mathrm{cone}(w_1, w_4)^\circ$ we infer $\mu_1, \mu_2 > 0$ and thus conclude $\mu_1 = \mu_2 = 1$. In particular, we have $w_2 = (0,1)$, $w_3 = (1,0)$ and $\tau_X = \mathbb{Q}^2_{\geq 0}$. Moreover, $\mu = (1,1)$ implies that r is even. Suitably renumbering the T_i with $i \geq 5$, we achieve $w_i \in \tau^-$ and $w_{i+1} \in \tau^+$ for $i = 5, 7, \ldots, r-1$. Then, for every odd i, Remark 3.2.5 and homogeneity of g provide us with the conditions

$$\det(w_i, w_2) \ = \ 1\,, \qquad w_i + w_{i+1} \ = \ \mu = (1,1)\,, \qquad \det(w_1, w_{i+1}) \ = \ 1\,.$$

We conclude $w_i = (1,0)$ and $w_{i+1} = (0,1)$ for all $i = 5, 7, \ldots, n-1$. The weights $u_j = \deg(S_j)$ are contained either all in τ^- or all in τ^+. We may assume that all lie in τ^+. Applying Remark 3.2.5 to $\gamma_{1,r+j}$, where $j = 1, \ldots, t$, yields $u_j = (a_j, 1)$ with some $a_j \in \mathbb{Z}_{\leq 0}$. A suitable linear coordinate change in $\mathbb{Z}^2$ leads to Setting 4.

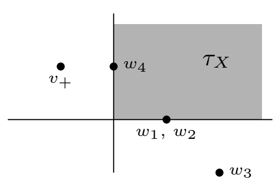
Case (ii): Here we have $\tau_X = \mathrm{cone}(w_1, w_4)$, $w_1 \in \tau^-$ and $w_4 \in \tau^+$. Applying Remark 3.2.5 to $\gamma_{1,4}$ yields $w_1 = (1,0)$ and $w_4 = (0,1)$. We distinguish the two subcases $w_2 \in \tau^-$, $w_3 \in \tau^-$ and $w_2 \in \tau^+$, $w_3 \in \tau^-$.

In the latter subcase, i.e. if $w_2 \in \tau^+$ and $w_3 \in \tau^-$ hold, we proceed exactly as in Case (i) and thus arrive in Setting 4.

If $w_2 \in \tau^-$ holds, we conclude that μ lies in τ^-. Hence the same holds for w_r if r is odd. Let v_+ be any weight in τ^+. Applying Remark 3.2.5 to (w_1, v_+) yields $v_+^2 = 1$. The same remark applied to γ_{24} shows $w_2^1 = 1$ and thus $\mu_1 = 2$. The homogeneity of g yields $w_3^1 = 2$. We now apply Remark 3.2.5 to (w_2, v_+) and (w_3, v_+). Thus, we obtain

$$1 \ = \ 1 - w_2^2 v_+^1 \qquad \text{and} \qquad 1 \ = \ 2 - w_2^2 v_+^1 + v_+^1\,,$$

where we used $w_3 = w_1 + w_2 - w_4$ for the last equality. We conclude $w_2^2 = 0$, $v_+^1 = -1$ and $w_3^2 = -1$, i.e. the situation is as follows:



If r is odd, then $\mu = (2,0)$ shows that $w_r = (1,0)$ holds. Now consider an odd integer $5 \leq i < r$. Since $\mu \in \tau^-$ holds, we may assume that $w_i \in \tau^-$ holds. Remark 3.2.5 first applied to $\gamma_{4,i}$ and then to (w_i, v^+) shows that $w_i = (1,0)$ holds. The homogeneity of g yields $w_{i+1} = (1,0)$. Thus we conclude that $w_\ell = (1,0)$ holds for all $5 \leq \ell \leq r$. In particular, v_+ is of type u_j. This means that we may assume that $u_1 = v_+$ holds. Let $2 \le j \le t$. Lemma 3.1.5 shows that $u_j \in \tau^+$ holds. Remark 3.2.5 first applied to (w_1, u_j) and then to (w_3, u_1) yields $u_j = u_1$. After renumbering the variables and multiplying the degree matrix Q with some unimodular matrix from the left, we arrive in Setting 3.

Case (iii): Here we have $\tau_X = \operatorname{cone}(u_1, w_2)$, $u_1 \in \tau^-$ and $w_2 \in \tau^+$. Applying Remark 3.2.5 to (u_1, w_2) yields $u_1 = (1,0)$ and $w_2 = (0,1)$. We distinguish two subcases $w_1 \in \tau^+$ and $w_1 \in \tau^-$.

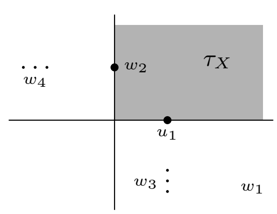
In the first case, we have $w_1 \in \tau^+$ and thus $\mu \in \tau^+$. Hence we may assume that all w_i, i odd, are contained in τ^+. Remark 3.2.5 applied to (u_1, w_i), i odd, $i \neq r$, shows that $w_i^2 = 1$ holds for all odd $i \neq r$. In particular, we have $\mu = w_1 + w_2 = (w_1^1, 2)$. Consider an odd index $i \neq r$. Homogeneity of g yields $w_{i+1}^2 = 1$. In particular, w_{i+1} is contained in τ^+. Hence all weights of type w_i are contained in τ^+ and thus we have $t \geq 2$. Now consider a weight u_j, $2 \le j \le t$. Lemma 3.1.5 shows that u_j is contained in τ^-. Together with Remark 3.2.5 and (u_j, w_2), we obtain $u_j^1 = 1$. Now the same remark applied to all pairs (w_i, u_j) shows that we have $w_i = (0,1)$ for all $1 \le i \le r$ or $u_j = (1,0)$ for all $1 \le j \le t$. Multiplying with some invertible integer matrix, we arrive in Setting 1 or 2.

Now we treat the case $w_1 \in \tau^-$. This means that we have $\mu \in \tau_X \cup \tau^-$. Hence we may assume that $w_3 \in \tau^-$ holds. Remark 3.2.5 applied to γ_{23} yields $w_3^1 = 1$. Note that the homogeneity of g yields

$$w_1 \;=\; (\mu_1, \mu_2 - 1) \quad \text{and} \quad w_4 \;=\; (\mu_1 - 1, \mu_2 - w_3^2)\,.$$

We show that $w_4 \in \tau^+$ holds. Indeed, assume that w_4 lies in τ^-. Then Remark 3.2.5 applied to γ_{24} shows that $w_4^1 = 1$ holds. Thus we have $\mu_1 = 2$ and $w_1 = (2, \mu_2 - 1)$. Let $w_2 \neq v_+$ be a weight in τ^+. Since $u_1 \in \tau^-$ holds, Lemma 3.1.5 shows that v_+ is of type w_i. Note that $\mu = w_3 + w_4 \in \tau^-$ holds and thus w_r is contained in τ^- if r is odd. This means that we may assume that $v_+ = w_5$ and $w_6 \in \tau^-$ hold. We apply Remark 3.2.5 firstly to (u_1, w_5) and then to (w_1, w_5) and arrive at $w_5^2 = 1$ and $1 = 2 - w_1^2 w_5^1$. Since $w_5^1 \le 0$ holds, we conclude $w_5^1 = w_1^2 = -1$. Homogeneity of g yields $w_6^1 = \mu_1 - w_5^1 = 3$. But then Remark 3.2.5 applied to γ_{26} yields $1 = \det(w_6, w_2) = 3$, a contradiction.

Hence we have $w_4 \in \tau^+$. Remark 3.2.5 applied to (u_1, w_4) yields $w_4^2 = 1$. Thus, the situation is as follows:



Since $w_1 = (\mu_1, \mu_2 - 1)$ lies in τ^-, we have $\mu_1 \geq 1$. But $w_4 = (\mu_1 - 1, 1) \in \tau^+$ yields $\mu_1 - 1 \le 0$. Together, we obtain $\mu_1 = 1$. In particular, μ is primitive and thus r is even. Furthermore, we have $w_4 = w_2 = (0,1)$ and $w_1 = w_3 = (1, \mu_2 - 1)$. Note that Remark 3.2.5 applied to (w_2, w_i) yields $w_i^1 = 1$ for all $w_i \in \tau^-$, $5 \le i \le r$. Moreover, we have $w_\ell^1 \le 0$ for all $w_\ell \in \tau^+$. Because of $1 = \mu_1 = w_i^1 + w_{i+1}^1$ for all odd $5 \le i \le r$, renumbering of variables yields $w_i = (1, w_i^2) \in \tau^-$ and $w_{i+1} = (0, w_{i+1}^2) \in \tau^+$ for all odd $5 \le i \le r$. Remark 3.2.5 applied to (u_1, w_{i+1}) yields $w_{i+1} = (0,1)$ for all odd $5 \le i \le r$. Lemma 3.1.5 shows that all further weights of type u_j lie in τ^-. Applying Remark 3.2.5 to (u_j, w_2), and by multiplying with some unimodular matrix from the left, we arrive in Setting 4. $\square$

3.3. Classification results in Picard number three

In this section we state our classification results for smooth intrinsic quadrics of Picard number three. We first describe in Theorem 3.3.2 all full smooth intrinsic quadrics of Picard number three and arbitrary dimension. We conclude in particular that all full smooth Fano intrinsic quadrics have Picard number at most two, see Corollary 3.3.3. We then consider smooth intrinsic quadrics of Picard number three and dimension at most four, see Theorems 3.3.5 and 3.3.6. Moreover, we describe the (almost) Fano varieties in this setting, see Theorems 3.3.5, 3.3.8 and 3.3.10. The proofs are given in Sections 3.6 to 3.10.

Construction 3.3.1. Consider the $\mathbb{K}$-algebra $R = \mathbb{K}[T_1, \ldots, T_r]/\langle g \rangle$, where $g = T_1T_2 + \ldots + T_{r-1}T_r$ holds for some integer $r \in \mathbb{Z}_{\geq 8}$. Define a $\mathbb{Z}^3$-grading on R by choosing weights $w_i = \deg(T_i)$ according to the following setting. The polynomial g is of degree $(0, 1, 1)$ and the weights are as follows:

(i) At least two monomials T_iT_{i+1} of g fulfill $w_i = (0, 1, 0)$, $w_{i+1} = (0, 0, 1)$.
(ii) At least two monomials T_iT_{i+1} of g fulfill $w_i = (1, a_i, 0)$ and $w_{i+1} = (-1, 1-a_i, 1)$ with some $a_i \in \mathbb{Z}_{\geq 0}$, where $(1, 0, 0)$ and $(-1, 1, 1)$ show up as degrees of variables.

Moreover, all monomials of g are as described in (i) or (ii). Set

$$\begin{aligned}\tau \quad := \quad & \operatorname{cone}\big((0,1,0),(0,0,1),(1,\max(a_i),0))\big) \\ & \cap \operatorname{cone}\big((-1,1,1),(0,1,0),(1,\max(a_i),0)\big)\,.\end{aligned}$$

The polynomial g is $\mathbb{Z}^3$-homogeneous and R is the Cox ring of a full smooth intrinsic quadric X with semiample cone $\tau \subseteq \mathbb{Q}^3$.

Theorem 3.3.2. *Let X be a full intrinsic quadric of Picard number three. If X is smooth, then X is isomorphic to an intrinsic quadric arising from Construction 3.3.1.*

Corollary 3.3.3. *Let X be a smooth full intrinsic quadric. If X is Fano, then the Picard number of X is at most two. In particular, X then is either isomorphic to one of the varieties of Proposition 3.2.1 or to one of the intrinsic quadrics of Setting 4 in Theorem 3.2.10 with $\alpha = t = 0$.*

Proof. In Proposition 3.2.1 we gave a description of the smooth Fano full intrinsic quadrics of Picard number one. The smooth Fano full intrinsic quadrics of Picard number two follow from the classification in **[11]**; they are also listed in Theorem 3.2.10, Setting 4 with $\alpha = t = 0$. In Proposition 3.1.7 we proved that $\rho(X)$ is at most three if X is a smooth Fano intrinsic quadric. Thus it remains to show that there is no smooth Fano intrinsic quadric of Picard number three, i.e. that none of the intrinsic quadrics arising from Construction 3.3.1 is Fano. This can be seen as follows: Computing the anticanonical class $\mathcal{K}_X$ shows that $\mathcal{K}_X$ is a multiple of $\deg(g)$. In the setting of Construction 3.3.1, $\deg(g)$ does not lie in the relative interior of the cone τ, which completes the proof. □

Corollary 3.3.4. *Let X be a smooth Fano full intrinsic quadric. Then X fulfills Mukai's conjecture.*

Proof. This is an immediate consequence of Proposition 3.2.14 and Corollary 3.3.3. □

Theorem 3.3.5. *Every smooth intrinsic quadric of Picard number three and dimension at most three is isomorphic to one of the following varieties X, specified by their Cox ring $\mathcal{R}(X)$ and their semiample cone* $\operatorname{SAmple}(X)$, *where we always have* $\operatorname{Cl}(X) = \mathbb{Z}^3$ *and the grading is fixed by the matrix $Q = [w_1, \ldots, w_7]$ of generator* *degrees* $w_i = \deg(T_i) \in \mathrm{Cl}(X)$. *If not indicated otherwise, the letter* a *denotes an arbitrary integer.*

<table>
<tr><th>No.</th><th>$\mathcal{R}(X)$</th><th>$Q = [w_1, \ldots, w_7]$</th><th>$\mathrm{SAmple}(X)$ is the intersection of the following cones</th></tr>
<tr><td>1</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_7]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$</td><td>$\left[\begin{array}{cc|cc|cc||c} 1 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & a \end{array}\right]$
$a < 0$</td><td>$\mathrm{cone}(w_1, w_3, w_5)$, $\mathrm{cone}(w_2, w_5, w_7)$</td></tr>
<tr><td>2</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_7]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$</td><td>$\left[\begin{array}{cc|cc|cc||c} 1 & a-1 & 0 & a & 0 & a & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{array}\right]$
$a > 0$</td><td>$\mathrm{cone}(w_1, w_4, w_6)$, $\mathrm{cone}(w_2, w_3, w_7)$</td></tr>
<tr><td>3</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_7]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$</td><td>$\left[\begin{array}{cc|cc|c||cc} 1 & 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & -1 \\ 0 & a & a & 0 & a/2 & 1 & 1 \end{array}\right]$
$a \geq 0$</td><td>$\mathrm{cone}(w_2, w_3, w_6)$, $\mathrm{cone}(w_2, w_6, w_7)$</td></tr>
<tr><td>4</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_7]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$</td><td>$\left[\begin{array}{cc|cc|c||cc} 1 & 1 & 2 & 0 & 1 & a & 0 \\ 0 & 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$</td><td>$\mathrm{cone}(w_1, w_3, w_6)$, $\mathrm{cone}(w_1, w_4, w_7)$</td></tr>
<tr><td>5</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_7]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$</td><td>$\left[\begin{array}{cc|cc|c||cc} 1 & 1 & 2 & 0 & 1 & -2 & 0 \\ 0 & 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & a & a & 0 & a/2 & 1-a & 1 \end{array}\right]$
$a > 0$</td><td>$\mathrm{cone}(w_2, w_4, w_7)$, $\mathrm{cone}(w_2, w_6, w_7)$</td></tr>
<tr><td>6</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_7]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$</td><td>$\left[\begin{array}{cc|cc|c||cc} 1 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & -1 \\ 0 & -2 & 0 & -2 & -1 & 1 & 1 \end{array}\right]$</td><td>$\mathrm{cone}(w_1, w_3, w_6)$, $\mathrm{cone}(w_1, w_3, w_7)$</td></tr>
</table>

Moreover, each of the listed data sets defines a smooth intrinsic quadric of Picard number three and dimension three. The Fano varieties among the varieties listed in the above table are exactly the following varieties:

No. 1 with $a = -1$, *No. 3 with* $a = 0$ *and No. 4 with* $-2 \leq a \leq 0$.

The truly almost Fano varieties among the varieties listed in the above table are exactly the following varieties:

No. 1 with $a \in \{-2, 0\}$ *and No. 4 with* $a \in \{-3, 1\}$.

Note that all smooth intrinsic quadrics of Picard number and dimension three have Fano index one. We now turn to our classification results in dimension four.

Theorem 3.3.6. *Every smooth intrinsic quadric of Picard number three and dimension four is isomorphic to one of the following varieties* X, *specified by their Cox ring* $\mathcal{R}(X)$ *and their semiample cone* $\mathrm{SAmple}(X)$, *where we always have* $\mathrm{Cl}(X) = \mathbb{Z}^3$ *and the grading is fixed by the matrix* $Q = [w_1, \ldots, w_8]$ *of generator degrees* $w_i = \deg(T_i) \in \mathrm{Cl}(X)$. *If not indicated otherwise, the letters* a, b *and* c *denote arbitrary integers.*

<table>
<tr><th>No.</th><th>$\mathcal{R}(X)$</th><th>$Q = [w_1, \ldots, w_8]$</th><th>$\mathrm{SAmple}(X)$ is the the intersection of the following cones</th></tr>
<tr><td>1</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7T_8\rangle}$</td><td>$\left[\begin{array}{cc|cc|cc|cc} 1 & a-1 & 0 & a & 0 & a & 1 & a-1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$
$a \geq 0$</td><td>$\mathrm{cone}(w_1, w_6, w_4 + w_6)$</td></tr>
<tr><td>2</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$</td><td>$\left[\begin{array}{cc|cc|cc||cc} 1 & -1 & 0 & 0 & 0 & 0 & a & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & b & c \end{array}\right]$
$b \leq 0,\ c < 0$</td><td>$\mathrm{cone}(w_1, w_3, w_5)$, $\mathrm{cone}(w_1, w_5, w_7)$,
$\mathrm{cone}(w_2, w_5, w_8)$, $\mathrm{cone}(w_4, w_7, w_8)$</td></tr>
<tr><td>3</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,S_2]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$</td><td>$\left[\begin{array}{cc|cc|cc||cc} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & a \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right]$</td><td>$\mathrm{cone}(w_1, w_5, w_7)$, $\mathrm{cone}(w_2, w_3, w_8)$</td></tr>
<tr><td>4</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$</td><td>$\left[\begin{array}{cc|cc|cc||cc} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 & 1 & 0 & a & 0 \end{array}\right]$</td><td>$\mathrm{cone}(w_1, w_3, w_4)$, $\mathrm{cone}(w_2, w_7, w_8)$</td></tr>
</table>

<table>
<tr><td>5</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$</td><td>$\left[\begin{array}{cc|cc|cc||cc} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & -2 \\ 0 & 1 & 0 & 1 & 1 & 0 & -1 & 1 \end{array}\right]$</td><td>$\mathrm{cone}(w_1,w_5,w_7)$, $\mathrm{cone}(w_1,w_6,w_8)$</td></tr>
<tr><td>6</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$</td><td>$\left[\begin{array}{cc|cc|cc||cc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & a & b \end{array}\right]$
$0 > a \geq b$</td><td>$\mathrm{cone}(w_1,w_3,w_5)$, $\mathrm{cone}(w_2,w_5,w_7)$</td></tr>
<tr><td>7</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$</td><td>$\left[\begin{array}{cc|cc|cc||cc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 1 & 0 & a & 0 \end{array}\right]$
$a < 0$</td><td>$\mathrm{cone}(w_1,w_3,w_5)$</td></tr>
<tr><td>8</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$</td><td>$\left[\begin{array}{cc|cc|cc||cc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & a \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & -1 & 1 \end{array}\right]$</td><td>$\mathrm{cone}(w_1,w_3,w_5)$, $\mathrm{cone}(w_1,w_7,w_8)$</td></tr>
<tr><td>9</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$</td><td>$\left[\begin{array}{cc|cc|cc||cc} 1 & a-1 & 0 & a & 0 & a & b & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right]$
$a \geq 0$</td><td>$\mathrm{cone}(w_1,w_4,w_6)$, $\mathrm{cone}(w_2,w_6,w_8)$, $\mathrm{cone}(w_4,w_7,w_8)$</td></tr>
<tr><td>10</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,S_2]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$</td><td>$\left[\begin{array}{cc|cc|cc||cc} 1 & a-1 & 0 & a & 0 & a & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right]$
$a > 0$</td><td>$\mathrm{cone}(w_1,w_4,w_6)$, $\mathrm{cone}(w_2,w_6,w_7)$</td></tr>
<tr><td>11</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$</td><td>$\left[\begin{array}{cc|cc|cc||cc} 1 & a-1 & 0 & a & 0 & a & 1 & b \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & c \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$
$a \geq 0$</td><td>$\mathrm{cone}(w_1,w_6,w_8)$, $\mathrm{cone}(w_2,w_6,w_7)$, $\mathrm{cone}(w_4,w_6,w_7)$</td></tr>
<tr><td>12</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$</td><td>$\left[\begin{array}{cc|cc|cc||cc} 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & a & 0 \\ 0 & -1 & 0 & -1 & 0 & -1 & 1 & 1 \end{array}\right]$</td><td>$\mathrm{cone}(w_2,w_3,w_7)$, $\mathrm{cone}(w_1,w_3,w_8)$</td></tr>
<tr><td>13</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$</td><td>$\left[\begin{array}{cc|cc|cc||cc} 1 & 0 & 0 & 1 & 1 & 0 & a & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$</td><td>$\mathrm{cone}(w_1,w_3,w_7)$, $\mathrm{cone}(w_1,w_3,w_8)$</td></tr>
<tr><td>14</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$</td><td>$\left[\begin{array}{cc|cc|cc||cc} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & a & 0 & a & b & a-b & 1 & 1 \end{array}\right]$</td><td>$\mathrm{cone}(w_1,w_6,w_7)$, $\mathrm{cone}(w_2,w_4,w_7)$, $\mathrm{cone}(w_2,w_5,w_7)$, $\mathrm{cone}(w_3,w_5,w_7)$, $\mathrm{cone}(w_4,w_6,w_7)$, $\mathrm{cone}(w_1,w_3,w_7)$</td></tr>
<tr><td>15</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$</td><td>$\left[\begin{array}{cc|cc|cc||cc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 & 1 & 1 & -1 & 0 \\ 0 & -2 & 0 & -2 & -1 & -1 & 1 & 1 \end{array}\right]$</td><td>$\mathrm{cone}(w_1,w_3,w_7)$, $\mathrm{cone}(w_1,w_3,w_8)$</td></tr>
<tr><td>16</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$</td><td>$\left[\begin{array}{cc|cc|cc||cc} 1 & 1 & 0 & 2 & 1 & 1 & 0 & 0 \\ 0 & a & 0 & a & b & a-b & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \end{array}\right]$</td><td>$\mathrm{cone}(w_1,w_4,w_8)$, $\mathrm{cone}(w_1,w_7,w_8)$, $\mathrm{cone}(w_2,w_4,w_8)$, $\mathrm{cone}(w_2,w_7,w_8)$, $\mathrm{cone}(w_5,w_4,w_8)$, $\mathrm{cone}(w_5,w_7,w_8)$, $\mathrm{cone}(w_6,w_4,w_8)$, $\mathrm{cone}(w_6,w_7,w_8)$</td></tr>
<tr><td>17</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$</td><td>$\left[\begin{array}{cc|cc|cc||cc} 1 & 1 & 0 & 2 & 1 & 1 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$</td><td>$\mathrm{cone}(w_1,w_3,w_7)$, $\mathrm{cone}(w_1,w_4,w_8)$, $\mathrm{cone}(w_1,w_7,w_8)$</td></tr>
<tr><td>18</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$</td><td>$\left[\begin{array}{cc|cc|cc||cc} 1 & 1 & 0 & 2 & 1 & 1 & 0 & -2 \\ a & 1 & 0 & 1+a & a+b & 1-b & 1 & -a \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$</td><td>$\mathrm{cone}(w_1,w_3,w_7)$, $\mathrm{cone}(w_1,w_7,w_8)$, $\mathrm{cone}(w_2,w_3,w_7)$, $\mathrm{cone}(w_2,w_7,w_8)$, $\mathrm{cone}(w_5,w_3,w_7)$, $\mathrm{cone}(w_5,w_7,w_8)$, $\mathrm{cone}(w_6,w_3,w_7)$, $\mathrm{cone}(w_6,w_7,w_8)$</td></tr>
<tr><td>19</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$</td><td>$\left[\begin{array}{cc|cc|c||ccc} 1 & -1 & 0 & 0 & 0 & 0 & a & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & b & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$</td><td>$\mathrm{cone}(w_1,w_3,w_6)$, $\mathrm{cone}(w_1,w_3,w_7)$</td></tr>
<tr><td>20</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$</td><td>$\left[\begin{array}{cc|cc|c||ccc} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & a & b \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right]$</td><td>$\mathrm{cone}(w_1,w_3,w_6)$, $\mathrm{cone}(w_1,w_3,w_7)$, $\mathrm{cone}(w_2,w_3,w_8)$, $\mathrm{cone}(w_3,w_7,w_8)$</td></tr>
<tr><td>21</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$</td><td>$\left[\begin{array}{cc|cc|c||ccc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & a & b \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right]$
$a \geq b$</td><td>$\mathrm{cone}(w_1,w_3,w_6)$, $\mathrm{cone}(w_2,w_3,w_7)$, $\mathrm{cone}(w_3,w_6,w_7)$</td></tr>
<tr><td>22</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$</td><td>$\left[\begin{array}{cc|cc|c||ccc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & a & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$
$a \neq -2$</td><td>$\mathrm{cone}(w_1,w_3,w_6)$, $\mathrm{cone}(w_2,w_3,w_7)$</td></tr>
<tr><td>23</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$</td><td>$\left[\begin{array}{cc|cc|c||ccc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & a-1 \\ 0 & 2 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & a \end{array}\right]$</td><td>$\mathrm{cone}(w_1,w_7,w_8)$, $\mathrm{cone}(w_2,w_3,w_7)$, $\mathrm{cone}(w_3,w_6,w_7)$, $\mathrm{cone}(w_3,w_7,w_8)$</td></tr>
<tr><td>24</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$</td><td>$\left[\begin{array}{cc|cc|c||ccc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 2 & 1 & 1 & 1 & 0 & a & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$</td><td>$\mathrm{cone}(w_1,w_3,w_6)$, $\mathrm{cone}(w_2,w_3,w_7)$</td></tr>
</table>

25	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|ccc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & a \\ 0 & 2 & 1 & 1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right]$	$\mathrm{cone}(w_1,w_3,w_6)$, $\mathrm{cone}(w_1,w_7,w_8)$
26	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|ccc} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & 0 & -1 \\ 0 & a & 0 & a & a/2 & 1 & 1 & b \end{array}\right]$ $a \in 2\mathbb{Z},\ a \le 0$	$\mathrm{cone}(w_1,w_3,w_7)$, $\mathrm{cone}(w_2,w_6,w_8)$, $\mathrm{cone}(w_3,w_6,w_8)$
27	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|ccc} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & 0 & -2 \\ 0 & a & 0 & a & a/2 & 1 & 1 & 1-a \end{array}\right]$ $a \in 2\mathbb{Z},\ a < 0$	$\mathrm{cone}(w_1,w_3,w_6)$, $\mathrm{cone}(w_3,w_6,w_8)$
28	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|ccc} 1 & -1 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & a & 0 & a & a/2 & 1 & 1 & 1 \end{array}\right]$ $a \in 2\mathbb{Z},\ a < 0$	$\mathrm{cone}(w_2,w_3,w_8)$, $\mathrm{cone}(w_3,w_6,w_8)$
29	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|ccc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & -2 & -2 \\ 0 & a & 0 & a & a/2 & 1 & 1-a & 1-a \end{array}\right]$ $a \in 2\mathbb{Z},\ a \le 0$	$\mathrm{cone}(w_1,w_3,w_6)$, $\mathrm{cone}(w_3,w_6,w_7)$
30	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|ccc} 1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & a & 0 & a & a/2 & 1 & 1 & 1 \end{array}\right]$ $a \in 2\mathbb{Z},\ a \le 0$	$\mathrm{cone}(w_2,w_3,w_7)$, $\mathrm{cone}(w_3,w_6,w_7)$
31	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|ccc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & -2 & -1 \\ 0 & a & 0 & a & a/2 & 1 & 1-a & b \end{array}\right]$ $a \in 2\mathbb{Z},\ a \le 0$	$\mathrm{cone}(w_1,w_3,w_6)$, $\mathrm{cone}(w_2,w_6,w_8)$, $\mathrm{cone}(w_3,w_6,w_7)$, $\mathrm{cone}(w_3,w_6,w_8)$
32	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|ccc} 1 & -1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 2 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & a & 0 & a & a/2 & 1 & 1 & b \end{array}\right]$ $a \in 2\mathbb{Z},\ a \le 0$	$\mathrm{cone}(w_1,w_7,w_8)$, $\mathrm{cone}(w_2,w_3,w_7)$, $\mathrm{cone}(w_3,w_6,w_7)$, $\mathrm{cone}(w_3,w_7,w_8)$
33	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|ccc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 2 & 1 & 1 & 1 & 0 & -1 & 1 \\ 0 & a & 0 & a & a/2 & 1 & b & 1-b \end{array}\right]$ $a \in 2\mathbb{Z},\ a \le 0$	$\mathrm{cone}(w_1,w_3,w_6)$, $\mathrm{cone}(w_1,w_6,w_8)$, $\mathrm{cone}(w_2,w_7,w_8)$, $\mathrm{cone}(w_3,w_6,w_7)$
34	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|ccc} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & 0 & -1 \\ 0 & -2 & 0 & -2 & -1 & 1 & 1 & 1 \end{array}\right]$	$\mathrm{cone}(w_1,w_3,w_6)$, $\mathrm{cone}(w_1,w_3,w_8)$
35	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|ccc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & -1 & -1 \\ 0 & -2 & 0 & -2 & -1 & 1 & 1 & 1 \end{array}\right]$	$\mathrm{cone}(w_1,w_3,w_6)$, $\mathrm{cone}(w_1,w_3,w_8)$
36	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|ccc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & -1 & -1 \\ 0 & -2 & 0 & -2 & -1 & 1 & 1 & 0 \end{array}\right]$	$\mathrm{cone}(w_1,w_3,w_6)$, $\mathrm{cone}(w_1,w_3,w_7)$
37	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|ccc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 2 & 1 & 1 & 1 & 0 & -1 & 1 \\ 0 & -2 & 0 & -2 & -1 & 1 & 1 & 0 \end{array}\right]$	$\mathrm{cone}(w_1,w_3,w_6)$, $\mathrm{cone}(w_1,w_3,w_7)$

Moreover, each of the listed data sets defines a smooth intrinsic quadric of Picard number three and dimension four.

Remark 3.3.7. Note that some of the data sets listed in the table of Theorem 3.3.6 define isomorphic varieties; for instance No. 3 with $a = -1$ and No. 4 with $a = 0$, No. 3 with $a = 0$ and No. 9 with $a = b = 0$, or No. 19 with $a = 0$, $b := c \in \mathbb{Z}$ and No. 20 with $b = -1$, $a := c$. Moreover, there are non-isomorphic varieties sharing the same Cox ring; for instance No. 19 with $a = 1$, $b := c \in \mathbb{Z}$ and No. 22 with $a := c$ or No. 19 with $b = 1$ and No. 25.

Theorem 3.3.8. *Every smooth Fano intrinsic quadric of Picard number three and dimension four is isomorphic to one of the following varieties X, specified by their Cox ring $\mathcal{R}(X)$ and their semiample cone $\mathrm{SAmple}(X)$, where we always have $\mathrm{Cl}(X) = \mathbb{Z}^3$ and the grading is fixed by the matrix $Q = [w_1, \dots, w_8]$ of generator degrees $w_i = \deg(T_i) \in \mathrm{Cl}(X)$.*

No.	$\mathcal{R}(X)$	$Q = [w_1, \dots, w_8]$	$-\mathcal{K}_X$

2	$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & -1 \end{array}\right]$	$\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$
3	$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,S_2]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & a \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right]$ $-2 \le a \le 0$	$\begin{bmatrix} 1 \\ 3+a \\ 2 \end{bmatrix}$
4	$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 & 1 & 0 & a & 0 \end{array}\right]$ $-1 \le a \le 0$	$\begin{bmatrix} 1 \\ 2 \\ 2+a \end{bmatrix}$
7	$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 1 & 0 & -1 & 0 \end{array}\right]$	$\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$
9	$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right]$	$\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$
13, 14	$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & 0 & 0 & 1 & 1 & 0 & a & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$ $-1 \le a, b \le 1$	$\begin{bmatrix} 2+a \\ 2+b \\ 2 \end{bmatrix}$
16	$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & 1 & 0 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \end{array}\right]$	$\begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}$
17, 18	$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & 1 & 0 & 2 & 1 & 1 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$ $-3 \le a \le 1$	$\begin{bmatrix} 4+a \\ 2 \\ 1 \end{bmatrix}$
19	$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|ccc} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & a & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$ $-1 \le a \le 1$	$\begin{bmatrix} 1 \\ 2+a \\ 2 \end{bmatrix}$
20, 21, 30	$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|ccc} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right]$ $-2 \le a \le -1$	$\begin{bmatrix} 1 \\ 3+a \\ 3 \end{bmatrix}$
26	$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|ccc} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & a \end{array}\right]$ $-1 \le a \le 0$	$\begin{bmatrix} 1 \\ 2 \\ 2+a \end{bmatrix}$

Moreover, each of the listed data sets defines a smooth Fano intrinsic quadric of Picard number three and dimension four.

Remark 3.3.9. Note that the Fano intrinsic quadrics No. 3 with $a = -1$ and No. 4 with $a = 0$ coincide. The same holds for the Fano varieties No. 19 with $a = 0$ and No. 26 with $a = 0$. Hence there are up to isomorphism altogether 28 smooth Fano intrinsic quadrics of Picard number three and dimension four. Variety No. 13 with $a = b = 0$ has Fano index two and all other varieties of Theorem 3.3.8 have Fano index one.

Theorem 3.3.10. *Every smooth truly almost Fano intrinsic quadric of Picard number three and dimension four is isomorphic to one of the following varieties X, specified by their Cox ring $\mathcal{R}(X)$ and their semiample cone* $\mathrm{SAmple}(X)$*, where we always have* $\mathrm{Cl}(X) = \mathbb{Z}^3$ *and the grading is fixed by the matrix $Q = [w_1, \ldots, w_8]$ of generator degrees $w_i = \deg(T_i) \in \mathrm{Cl}(X)$.*

No.	$\mathcal{R}(X)$	$Q = [w_1, \ldots, w_8]$	$\mathrm{SAmple}(X)$ *is the intersection of the following cones*
1	$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7T_8\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|cc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$	$\mathrm{cone}(w_1, w_6, w_4 + w_6)$

2	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & -1 & 0 & 0 & 0 & 0 & a & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & b & c \end{array}\right]$ $-1 \leq a \leq 0,\ b=-1, c=-1$ $or\ -1 \leq a \leq 0,\ b=0, c=-2$ $or\ -1 \leq a \leq 0,\ b=1, c=0$ $or\ a=-1, b=0, c=-1$	$\mathrm{cone}(w_1,w_3,w_5)$, $\mathrm{cone}(w_1,w_5,w_7)$, $\mathrm{cone}(w_2,w_5,w_8)$, $\mathrm{cone}(w_4,w_7,w_8)$
3	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,S_2]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & -3 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right]$	$\mathrm{cone}(w_1,w_5,w_7)$, $\mathrm{cone}(w_2,w_3,w_8)$
4	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 & 1 & 0 & a & 0 \end{array}\right]$ $a=1\ or\ a=-2$	$\mathrm{cone}(w_1,w_3,w_4)$, $\mathrm{cone}(w_2,w_7,w_8)$
6	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & -1 & -1 \end{array}\right]$	$\mathrm{cone}(w_1,w_3,w_5)$, $\mathrm{cone}(w_2,w_5,w_7)$
7	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 1 & 0 & -2 & 0 \end{array}\right]$	$\mathrm{cone}(w_1,w_3,w_5)$
8	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & a \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & -1 & 1 \end{array}\right]$ $-1 \leq a \leq 0$	$\mathrm{cone}(w_1,w_3,w_5)$, $\mathrm{cone}(w_1,w_7,w_8)$
9	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & -1 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right]$	$\mathrm{cone}(w_1,w_4,w_6)$, $\mathrm{cone}(w_2,w_6,w_8)$, $\mathrm{cone}(w_4,w_7,w_8)$
10	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,S_2]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right]$	$\mathrm{cone}(w_1,w_4,w_6)$, $\mathrm{cone}(w_2,w_6,w_7)$
11	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & b \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$ $-1 \leq b \leq 0$	$\mathrm{cone}(w_1,w_6,w_8)$, $\mathrm{cone}(w_2,w_6,w_7)$, $\mathrm{cone}(w_4,w_6,w_7)$
12	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & a & 0 \\ 0 & -1 & 0 & -1 & 0 & -1 & 1 & 1 \end{array}\right]$ $-2 \leq a \leq -1$	$\mathrm{cone}(w_2,w_3,w_7)$, $\mathrm{cone}(w_1,w_3,w_8)$
13	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & 0 & 0 & 1 & 1 & 0 & a & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$ $a=\pm 2,\ -2 \leq b \leq 2$ $or\ b=\pm 2,\ -1 \leq a \leq 1$	$\mathrm{cone}(w_1,w_3,w_7)$, $\mathrm{cone}(w_1,w_3,w_8)$
14	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & b & -1-b & 1 & 1 \end{array}\right]$ $a=1,\ 0 \leq b \leq 1$ $or\ a=0,\ b=\pm 1$ $or\ a=-1,\ -1 \leq b \leq 0$	$\mathrm{cone}(w_1,w_6,w_7)$, $\mathrm{cone}(w_2,w_4,w_7)$, $\mathrm{cone}(w_2,w_5,w_7)$, $\mathrm{cone}(w_3,w_5,w_7)$, $\mathrm{cone}(w_4,w_6,w_7)$, $\mathrm{cone}(w_1,w_3,w_7)$
17	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cc\|cc\|cc\|\|cc} 1 & 1 & 0 & 2 & 1 & 1 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$ $a=-4\ or\ a=2$	$\mathrm{cone}(w_1,w_3,w_7)$, $\mathrm{cone}(w_1,w_4,w_8)$, $\mathrm{cone}(w_1,w_7,w_8)$
19	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|ccc} 1 & -1 & 0 & 0 & 0 & 0 & a & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & b & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$ $a=\pm 1,\ -2 \leq b \leq 2$ $or\ a=0, b=\pm 2$	$\mathrm{cone}(w_1,w_3,w_6)$, $\mathrm{cone}(w_1,w_3,w_7)$
20	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|ccc} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & a & b \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right]$ $a=-1,\ -2 \leq b \leq -1$ $or\ (a,b)=(0,-3)$ $or\ a=1,\ -1 \leq b \leq 0$	$\mathrm{cone}(w_1,w_3,w_6)$, $\mathrm{cone}(w_1,w_3,w_7)$, $\mathrm{cone}(w_2,w_3,w_8)$, $\mathrm{cone}(w_3,w_7,w_8)$
21	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{cc\|cc\|c\|\|ccc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & a & b \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right]$ $(a,b)=(-1,-2)$ $or\ (a,b)=(0,-1)$ $or\ (a,b)=(1,1)$	$\mathrm{cone}(w_1,w_3,w_6)$, $\mathrm{cone}(w_2,w_3,w_7)$, $\mathrm{cone}(w_3,w_6,w_7)$

22	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{rr\|rr\|r\|\|rrr} 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & a & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$ $-1 \leq a \leq 2$	$\mathrm{cone}(w_1, w_3, w_6)$, $\mathrm{cone}(w_2, w_3, w_7)$
23	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{rr\|rr\|r\|\|rrr} 1 & -1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 2 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$	$\mathrm{cone}(w_1, w_7, w_8)$, $\mathrm{cone}(w_2, w_3, w_7)$, $\mathrm{cone}(w_3, w_6, w_7)$, $\mathrm{cone}(w_3, w_7, w_8)$
24	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{rr\|rr\|r\|\|rrr} 1 & -1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 2 & 1 & 1 & 1 & 0 & a & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$ $-4 \leq a \leq 0$	$\mathrm{cone}(w_1, w_3, w_6)$, $\mathrm{cone}(w_2, w_3, w_7)$
26	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{rr\|rr\|r\|\|rrr} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & -2 \end{array}\right]$	$\mathrm{cone}(w_1, w_3, w_7)$, $\mathrm{cone}(w_2, w_6, w_8)$, $\mathrm{cone}(w_3, w_6, w_8)$
31	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{rr\|rr\|r\|\|rrr} 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$	$\mathrm{cone}(w_1, w_3, w_6)$, $\mathrm{cone}(w_2, w_6, w_8)$, $\mathrm{cone}(w_3, w_6, w_7)$, $\mathrm{cone}(w_3, w_6, w_8)$
32	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{rr\|rr\|r\|\|rrr} 1 & -1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 2 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$	$\mathrm{cone}(w_1, w_7, w_8)$, $\mathrm{cone}(w_2, w_3, w_7)$, $\mathrm{cone}(w_3, w_6, w_7)$, $\mathrm{cone}(w_3, w_7, w_8)$
34	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{rr\|rr\|r\|\|rrr} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & 0 & -1 \\ 0 & -2 & 0 & -2 & -1 & 1 & 1 & 1 \end{array}\right]$	$\mathrm{cone}(w_1, w_3, w_6)$, $\mathrm{cone}(w_1, w_3, w_8)$
35	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{rr\|rr\|r\|\|rrr} 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & -1 & -1 \\ 0 & -2 & 0 & -2 & -1 & 1 & 1 & 1 \end{array}\right]$	$\mathrm{cone}(w_1, w_3, w_6)$, $\mathrm{cone}(w_1, w_3, w_8)$

Moreover, each of the listed data sets defines a smooth truly almost Fano intrinsic quadric of Picard number three and dimension four.

Remark 3.3.11. As a consequence of Theorem 3.3.8, every smooth Fano intrinsic quadric of Picard number three and dimension four admits a torus action of complexity one and there is exactly one smooth truly almost Fano intrinsic quadric of Picard number three and dimension four, namely No. 1, that does not admit a torus action of complexity one.

Remark 3.3.12. Note that duplication of a free weight as introduced in Construction 2.2.1 yields many examples of higher dimensional intrinsic quadrics. In particular, all varieties arising via duplicating a free weight from one of the three-dimensional quadrics in the table of Theorem 3.3.5 turn up in the table of Theorem 3.3.8.

3.4. Geometry of the Fano intrinsic quadrics of Picard number three

In this section we take a closer look at the Fano varieties listed in Theorem 3.3.8 and describe explicitly their elementary birational divisorial contractions and their elementary contractions of fiber type.

Remark 3.4.1. We first give an overview which sort of elementary contraction is admitted by which smooth Fano intrinsic quadric of Picard number three. Since the Fano intrinsic quadrics No. 3 with $a = -1$ and No. 4 with $a = 0$ as well as the Fano varieties No. 19 with $a = 0$ and No. 26 with $a = 0$ coincide, we do not discuss the situations No. 4 with $a = 0$ and No. 26 with $a = 0$.

No.	birational divisorial, Y toric variety	birational divisorial, Y intrinsic quadric	fiber type	birational small
2	1	1	1	0
3, $-2 \leq a \leq -1$	1	1	1	0
3, $a = 0$	1	0	2	0
4, $a = -1$	1	2	0	0
7	1	2	0	0
9	1	0	1	1
13, $a = b = 0$	0	0	3	0

13, $a = b = \pm 1$	0	1	1	0
13, $a + b = \pm 1$	0	1	2	0
13, $a + b = 0$, $a, b \neq 0$	0	2	1	0
16	2	1	0	0
17, $a = -1$	2	2	0	0
17, $a \neq -1$	2	1	0	0
19, $a = \pm 1$	1	0	1	1
19, $a = 0$	1	1	1	0
20, $a = -1$	2	1	0	1
20, $a = -2$	2	1	0	0
26, $a = -1$	1	1	0	0

In the following, we describe explicitly the divisorial contractions and the contractions of fiber type listed in the above table.

No. 2: The variety X admits two birational divisorial contractions $\mathbb{P}_1 \times \mathbb{P}_3 \leftarrow X \rightarrow Y_2$, where Y_2 is a smooth intrinsic quadric from Setting 4 in Construction 3.2.7 with degree matrix and relation

$$Q = \left(\begin{array}{cc|cc|cc||c} 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \end{array}\right), \qquad g = T_1T_2 + T_3T_4 + T_5T_6.$$

The center of the two divisorial contractions are isomorphic to the intersection of a coordinate hypersurface with a divisor of bidegree $(1,1)$ and to $\mathbb{P}_2$, respectively. Furthermore, X admits a contraction of fiber type $X \rightarrow \mathbb{P}(\mathcal{O}_{\mathbb{P}_1} \oplus \mathcal{O}_{\mathbb{P}_1}(1))$.

No. 3, $\mathbf{-2 \leq a \leq -1}$: The variety X admits two birational divisorial contractions $Y_1 \leftarrow X \rightarrow Y_2$ and a contraction of fiber type $X \rightarrow Y_3$, where Y_3 is isomorphic to the projectivized split vector bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}_2} \oplus \mathcal{O}_{\mathbb{P}_2}(-a))$. If $a = -2$ holds, then we have

$$Y_1 \cong \mathbb{P}\left(\mathcal{O}_{\mathbb{P}_2} \oplus \bigoplus_{i=1}^{2} \mathcal{O}_{\mathbb{P}_2}(1)\right)$$

and if $a = -1$ holds, then we have $Y_1 \cong \mathbb{P}_2 \times \mathbb{P}_2$. In both cases, the center of $X \rightarrow Y_1$ is isomorphic to a divisor of bidegree (1,1) in $\mathbb{P}_1 \times \mathbb{P}_2$. If $a = -2$ holds, then Y_2 is isomorphic to a singular intrinsic quadric with degree matrix, defining relation and semiample cone given by

$$Q = \left(\begin{array}{cc|cc|cc||c} 2 & -1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{array}\right), \qquad g = T_1T_2 + T_3T_4 + T_5T_6$$

and $\mathrm{SAmple}(X) = \mathbb{Q}^2_{\geq 0}$. If $a = -1$ holds, then Y_2 is isomorphic to a smooth intrinsic quadric from Setting 4 in Construction 3.2.7 with degree matrix and relation

$$Q = \left(\begin{array}{cc|cc|cc||c} 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \end{array}\right), \qquad g = T_1T_2 + T_3T_4 + T_5T_6.$$

In both cases, the center of $X \rightarrow Y_2$ is isomorphic to $\mathbb{P}_1$.

No. 3, $\mathbf{a = 0}$: The variety X admits two contractions of fiber type $Y_1 \leftarrow X \rightarrow Y_2$ and one birational divisorial contraction $X \rightarrow Y_3$, where Y_3 is isomorphic to the projectivized split vector bundle

$$\mathbb{P}\left(\mathcal{O}_{\mathbb{P}_2} \oplus \bigoplus_{i=1}^{2} \mathcal{O}_{\mathbb{P}_2}(1)\right).$$

The center of this contraction is isomorphic to a divisor of bidegree (1,1) in the projectivized split vector bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}_1} \oplus \bigoplus_{i=1}^{2} \mathcal{O}_{\mathbb{P}_1}(1))$. Furthermore, Y_2 is isomorphic to $\mathbb{P}_1 \times \mathbb{P}_1$ and Y_3 is isomorphic to $\mathbb{P}_1 \times \mathbb{P}_2$.

No. 4, $\mathbf{a = -1}$: Here, the variety X admits three divisorial contractions $X \to Y_i$, $i = 1,2,3$. The variety Y_1 is isomorphic to the projectivized split vector bundle

$$\mathbb{P}\left(\mathcal{O}_{\mathbb{P}_2} \oplus \bigoplus_{i=1}^{2} \mathcal{O}_{\mathbb{P}_2}(1)\right)$$

and the center of the contraction $X \to Y_1$ is isomorphic to a divisor of bidegree $(1,1)$ in $\mathbb{P}(\mathcal{O}_{\mathbb{P}_1} \oplus \bigoplus_{i=1}^{2} \mathcal{O}_{\mathbb{P}_1}(1))$. The varieties Y_2 and Y_3 are smooth intrinsic quadrics from Construction 3.2.7. Y_2 belongs to Setting 4 of Construction 3.2.7 and has degree matrix and relation

$$Q = \left(\begin{array}{cc|cc|cc||c} 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \end{array}\right), \qquad g = T_1T_2 + T_3T_4 + T_5T_6.$$

The variety Y_3 belongs to Setting 3 of Construction 3.2.7 and has degree matrix and relation

$$Q = \left(\begin{array}{cc|cc|cc||c} 0 & 2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{array}\right), \qquad g = T_1T_2 + T_3T_4 + T_5T_6.$$

The centers of the contractions $X \to Y_2$ and $X \to Y_3$ are given by $\mathbb{P}_1$ and the intersection of the four prime divisors $D^1_{Y_3}$, $D^4_{Y_3}$, $D^5_{Y_3}$ and $D^7_{Y_3}$, respectively.

No. 7: Here, the variety X admits three divisorial contractions $\varphi_i\colon X \to Y_i$, $i = 1,2,3$. The variety Y_1 is isomorphic to $\mathbb{P}_2 \times \mathbb{P}_2$ with center isomorphic to a divisor of bidegree $(1,1)$ in $\mathbb{P}_1 \times \mathbb{P}_1$. The contractions φ_i, $i = 2,3$, are contractions from X to smooth intrinsic quadrics from Construction 3.2.7. Both Y_2 and Y_3 belong to Setting 4 of Construction 3.2.7 and have degree matrix and relation

$$Q = \left(\begin{array}{cc|cc|cc||c} 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \end{array}\right), \qquad g = T_1T_2 + T_3T_4 + T_5T_6.$$

The centers of the contractions $X \to Y_2$ and $X \to Y_3$ are both isomorphic to the first Hirzebruch surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}_1} \oplus \mathcal{O}_{\mathbb{P}_1}(1))$.

No. 9: Here, the variety X admits a divisorial contraction $\varphi_1\colon X \to Y_1$ and a contraction of fiber type $\varphi_2\colon X \to \mathbb{P}_1 \times \mathbb{P}_2$. The variety Y_1 is isomorphic to the projectivized split vector bundle

$$\mathbb{P}\left(\bigoplus_{i=1}^{2} \mathcal{O}_{\mathbb{P}_2} \oplus \mathcal{O}_{\mathbb{P}_2}(1)\right)$$

and the center of φ_1 is isomorphic to a divisor of bidegree $(1,1)$ in $\mathbb{P}_1 \times \mathbb{P}_2$.

No. 13, $\mathbf{a = b = 0}$: Here, the variety X is combinatorially minimal. It admits three contractions of fiber type $\varphi_i\colon X \to Y_i$, $i = 1,2,3$. The variety Y_1 is isomorphic to a divisor of bidegree $(1,1)$ in $\mathbb{P}_2 \times \mathbb{P}_2$ and we have $Y_2 \cong Y_3 \cong \mathbb{P}_2 \times \mathbb{P}_1$.

No. 13, $\mathbf{a = b = \pm 1}$: Here, the variety X admits a divisorial contraction $\varphi_1\colon X \to Y_1$ and a contraction of fiber type $\varphi_2\colon X \to Y_2$, where the variety Y_2 is isomorphic to a divisor of bidegree $(1,1)$ in $\mathbb{P}_2 \times \mathbb{P}_2$. The variety Y_1 is isomorphic to a non-$\mathbb{Q}$-factorial intrinsic quadric with degree matrix, defining relation and semiample cone given by

$$Q = \left(\begin{array}{cc|cc|cc||c} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{array}\right), \qquad g = T_1T_2 + T_3T_4 + T_5T_6$$

and $\mathrm{SAmple}(X) = \mathrm{cone}((1,1))$. The center of φ_1 is isomorphic to the intersection of the prime divisors $D^1_{Y_1}$, $D^4_{Y_1}$ and $D^5_{Y_1}$.

No. 13, $\mathbf{a + b = \pm 1}$: Here, the variety X admits a divisorial contraction $\varphi_1\colon X \to Y_1$ and two contractions of fiber type $\varphi_i\colon X \to Y_i$, $i = 2,3$, where Y_2 is isomorphic to a divisor of bidegree $(1,1)$ in $\mathbb{P}_2 \times \mathbb{P}_2$ and $Y_3 \cong \mathbb{P}_1 \times \mathbb{P}_2$ holds. The variety Y_1 is isomorphic to a smooth intrinsic quadric of Setting 4 from Construction 3.2.7 with degree matrix and relation

$$Q = \left(\begin{array}{cc|cc|cc||c} 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \end{array}\right), \qquad g = T_1T_2 + T_3T_4 + T_5T_6.$$

The center of the contraction $X \to Y_1$ is isomorphic to $\mathbb{P}_2$.

No. 13, $\mathbf{a+b=0}$, $\mathbf{a,b,\neq 0}$: Here, the variety X admits two divisorial contractions $\varphi_i\colon X \to Y_i$, $i = 1,2$ and a contraction of fiber type $\varphi_3\colon X \to Y_3$, where Y_3 is isomorphic to a divisor of bidegree $(1,1)$ in $\mathbb{P}_2 \times \mathbb{P}_2$. The varieties Y_1 and Y_2 are isomorphic to a smooth intrinsic quadric of Setting 4 from Construction 3.2.7 with degree matrix and relation

$$Q = \left(\begin{array}{cc|cc|cc||c} 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \end{array}\right), \qquad g = T_1T_2 + T_3T_4 + T_5T_6.$$

The centers of the contractions φ_i, $i = 1,2$ are isomorphic to $\mathbb{P}_2$.

No. 16: Here, the variety X admits three divisorial contractions $\varphi_i\colon X \to Y_i$, $i = 1,2,3$. The varieties Y_1 and Y_2 are both smooth toric varieties. To be precise, Y_1 is isomorphic to the projectivized split vector bundle

$$\mathbb{P}\left(\mathcal{O}_{\mathbb{P}_1} \oplus \bigoplus_{i=1}^{3} \mathcal{O}_{\mathbb{P}_1}(2)\right)$$

and the center of φ_1 is isomorphic to a divisor of bidegree $(1,1)$ in $\mathbb{P}(\mathcal{O}_{\mathbb{P}_1} \oplus \bigoplus_{i=1}^{2} \mathcal{O}_{\mathbb{P}_1}(2))$. The variety Y_2 is isomorphic to $\mathbb{P}_1 \times \mathbb{P}_3$ and the center of φ_2 is isomorphic to $\mathbb{P}_1 \times \mathbb{P}_1$. Furthermore, φ_3 is a contraction from X to a singular intrinsic quadric of Picard number two with degree matrix, defining relation and semiample cone given by

$$Q = \left(\begin{array}{cc|cc|cc||c} 1 & 1 & 0 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right), \qquad g = T_1T_2 + T_3T_4 + T_5T_6$$

and $\mathrm{SAmple}(X) = \mathbb{Q}^2_{\geq 0}$. The center of φ_3 is isomorphic to a point.

No. 17: Here, the variety X admits - depending on the value of a - three or four divisorial contractions $\varphi_i\colon X \to Y_i$, $i = 1,2,3,4$. For any choice of a, there are contractions φ_1 and φ_2 to smooth toric varieties. To be precise, Y_1 and Y_2 are isomorphic to the projectivized split vector bundles

$$Y_1 \cong \mathbb{P}\left(\mathcal{O}_{\mathbb{P}_3} \oplus \mathcal{O}_{\mathbb{P}_3}(|-a|)\right) \quad \text{and} \quad Y_2 \cong \mathbb{P}\left(\mathcal{O}_{\mathbb{P}_3} \oplus \mathcal{O}_{\mathbb{P}_3}(|a+2|)\right),$$

where $|x|$ denotes the absolute value of x. The centers of φ_1 and φ_2 are both isomorphic to $\mathbb{P}_1 \times \mathbb{P}_1$.

If $a \geq -1$ holds, then there is a further divisorial contraction $\varphi_3\colon X \to Y_3$, where Y_3 is an intrinsic quadric with degree matrix, defining relation and semiample cone given by

$$Q = \left(\begin{array}{cc|cc|cc||c} 1 & 1 & 0 & 2 & 1 & 1 & a \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right), \qquad g = T_1T_2 + T_3T_4 + T_5T_6$$

and semiample cone $\mathrm{SAmple}(X) = \mathbb{Q}^2_{\geq 0}$ if $-1 \leq a \leq 0$ holds, and $\mathrm{SAmple}(X) = \mathrm{cone}((1,0),(0,1))$ in case $a = 1$ holds. This means that Y_3 is smooth only if $a = -1$ holds. In this case, Y_3 belongs to Setting 3 of Construction 3.2.7. The center of φ_3 is isomorphic to a point if $a = 0$ holds, to the intersection of the prime divisors $D^i_{Y_3}$, $i = 1,2,3,5,6$ in case $a = -1$ holds, and to the intersection of the prime divisors $D^i_{Y_3}$, $i = 1,2,4,5,6$ if $a = 1$ holds.

If $a \le -1$ holds, then there is a further divisorial contraction $\varphi_4 \colon X \to Y_4$, where Y_4 is an intrinsic quadric with degree matrix, defining relation and semiample cone given by

$$Q = \left(\begin{array}{cc|cc|cc||c} 1 & 1 & 0 & 2 & 1 & 1 & -a \\ 1 & 1 & 1 & 1 & 1 & 1 & -a-1 \end{array}\right), \qquad g = T_1T_2 + T_3T_4 + T_5T_6$$

and $\mathrm{SAmple}(X) = \mathrm{cone}((1,1),(2,1))$ if $a = -2,-1$ holds, and $\mathrm{SAmple}(X) = \mathrm{cone}((1,1),(3,2))$ in case $a = -3$ holds. This means that Y_4 is smooth only if $a = -1$ holds. In this case, Y_4 belongs to Setting 3 of Construction 3.2.7. The center of φ_4 is isomorphic to a point if $a = -2$ holds, to the intersection of the prime divisors $D^i_{Y_4}$, $i = 1,2,4,5,6$ in case $a = -1$ holds, and to the intersection of the prime divisors $D^i_{Y_4}$, $i = 1,2,3,5,6$ in case $a = -3$ holds.

No. 19: Here, the variety X admits a divisorial contraction $\varphi_1 \colon X \to Y_1$ and a contraction of fiber type $\varphi_2 \colon X \to Y_2$, where Y_2 is isomorphic to a smooth intrinsic quadric of Setting 3 from Construction 3.2.7 with degree matrix and relation

$$Q = \left(\begin{array}{cc|cc|c||c} 0 & 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{array}\right), \qquad g = T_1T_2 + T_3T_4 + T_5^2\,.$$

The variety Y_1 is isomorphic to the projectivized split vector bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}_3} \oplus \mathcal{O}_{\mathbb{P}_3}(1))$ if $a = \pm 1$ holds and isomorphic to $\mathbb{P}_3 \times \mathbb{P}_1$ if $a = 0$ holds. The center of φ_1 is isomorphic to the first Hirzebruch surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}_1} \oplus \mathcal{O}_{\mathbb{P}_1}(1))$ if $a = \pm 1$ holds and isomorphic to $\mathbb{P}_1 \times \mathbb{P}_1$ if $a = 0$ holds.

If a equals zero, then we have a further divisorial contraction $\varphi_3 \colon X \to Y_3$, where Y_3 is isomorphic to a smooth intrinsic quadric of Setting 1 from Construction 3.2.7 with degree matrix and relation

$$Q = \left(\begin{array}{cc|cc|c||cc} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right), \qquad g = T_1T_2 + T_3T_4 + T_5^2$$

and center isomorphic to $\mathbb{P}_1$.

No. 20: Here, the variety X admits three divisorial contractions $\varphi_i \colon X \to Y_i$, $i = 1,2,3$. The varieties Y_1 and Y_2 are toric varieties and Y_3 is an intrinsic quadric. To be precise, Y_1 is isomorphic to the projectivized split vector bundle

$$\mathbb{P}\Big(\bigoplus_{i=1}^{2} \mathcal{O}_{\mathbb{P}_2} \oplus \mathcal{O}_{\mathbb{P}_2}(1)\Big)$$

if $a = -1$ holds and isomorphic to $\mathbb{P}_2 \times \mathbb{P}_2$ if $a = -2$ holds. In both cases, the center of φ_1 is isomorphic to $\mathbb{P}_1 \times \mathbb{P}_1$. The variety Y_2 is isomorphic to the projectivized split vector bundle

$$\mathbb{P}\Big(\mathcal{O}_{\mathbb{P}_2} \oplus \bigoplus_{i=1}^{2} \mathcal{O}_{\mathbb{P}_2}(-a)\Big)$$

and the center of φ_2 is isomorphic to $\mathbb{P}_1$. If $a = -1$ holds, then Y_3 is isomorphic to a smooth intrinsic quadric of Setting 3 from Construction 3.2.7 with degree matrix and relation

$$Q = \left(\begin{array}{cc|cc|c||cc} 0 & 2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{array}\right), \qquad g = T_1T_2 + T_3T_4 + T_5^2.$$

If $a = -2$ holds, then Y_3 is isomorphic to a singular intrinsic quadric with degree matrix, relation and semiample cone given by

$$Q = \left(\begin{array}{cc|cc|c||cc} 2 & 0 & 1 & 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 & 1 \end{array}\right), \qquad g = T_1T_2 + T_3T_4 + T_5^2$$

and $\mathrm{SAmple} = \mathbb{Q}^2_{\ge 0}$. In both cases, the center of φ_3 is isomorphic to $\mathbb{P}_1$.

No. 26, $\mathbf{a = -1}$: Here, the variety X admits two divisorial contractions $\varphi_i\colon X \to Y_i$, $i = 1, 2$, where Y_1 is toric and Y_2 is an intrinsic quadric. To be precise, Y_1 is isomorphic to the projectivized split vector bundle

$$\mathbb{P}\left(\mathcal{O}_{\mathbb{P}_1} \oplus \bigoplus_{i=1}^{3} \mathcal{O}_{\mathbb{P}_1}(1)\right)$$

and the center of φ_1 is isomorphic to $\mathbb{P}_1 \times \mathbb{P}_1$. The variety Y_2 is isomorphic to the smooth intrinsic quadric from Setting 3 of Construction 3.2.7 with degree matrix and relation

$$Q \;=\; \left(\begin{array}{cc|cc|c||cc} 0 & 2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{array}\right), \qquad g \;=\; T_1T_2 + T_3T_4 + T_5^2.$$

The center of φ_2 is isomorphic to the intersection of the prime divisors $D^2_{Y_2}$, $D^3_{Y_2}$, $D^4_{Y_2}$ and $D^5_{Y_2}$.

3.5. First structural constraints for Picard number three

In this section we consider intrinsic quadrics of Picard number three. We provide Lemmata we will need in the proofs in Sections 3.6 – 3.10. In particular, we show in Proposition 3.5.5, that the Picard group of a locally factorial intrinsic quadric of Picard number three is torsion-free.

Remark 3.5.1. In order to illustrate the arrangement of weights in $\mathrm{Cl}(X)_{\mathbb{Q}} = \mathbb{Q}^3$, we often choose a hypersurface H intersecting the effective cone in its relative interior and consider this two-dimensional picture. As a matter of convenience, we abbreviate $H \cap \mathrm{cone}(w_i)$ as w_i. What matters to us is which subsets of weights generate three-dimensional cones intersecting other cones in their relative interior. Thus for our purpose, the two-dimensional pictures only need to depict the position of a weight with respect to the position of the other weights and they do not need to be true to scale.

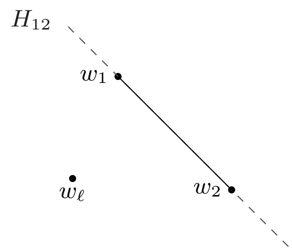
Setting 3.5.2. In the following, X is a standard intrinsic quadric of Picard number three. Hence its Cox ring is given by $\mathcal{R}(X) = \mathbb{K}[T_1, \ldots, T_r, S_1, \ldots, S_t]/\langle g\rangle$ with $g = T_1T_2 + \ldots + T_{q-1}T_q + h$ for some $0 \le q \le r$, $r \ge 3$, and some polynomial h given by

$$h \;=\; \begin{cases} T_{q+1}^2 + \ldots + T_r^2 & \text{if } q < r\,, \\ 0 & \text{if } q = r\,, \end{cases}$$

where we have $\deg(T_{q+k}) \neq \deg(T_{q+l})$ for all $1 \le k < l \le r - q$. By $w_i := \deg(T_i)$ and by $w_{r+j} := \deg(S_j)$, we denote the degrees of the variables T_i and S_j. By u we denote an ample Weil divisor class $u \in \mathrm{Cl}(X)$.

Lemma 3.5.3. *Let X be as in Setting 3.5.2 and assume that X is $\mathbb{Q}$-factorial. Assume that there is an index $5 \le \ell \le r + t$ such that T_ℓ is not a square. If $u \in Q(\gamma_{12\ell})^\circ$ and $g = T_1T_2 + T_3T_4 + \ldots$ hold, then we have $\gamma_{ij\ell} \in \mathrm{rlv}(u)$ for some $i \in \{1, 2\}$ and $j \in \{3, 4\}$.*

Proof. We denote by $l_{12} \in \mathrm{Hom}(\mathrm{Cl}(X), \mathbb{Q})$ a linear form with $l_{12}(w_1) = 0 = l_{12}(w_2)$ and $l_{12}(w_\ell) \ge 0$. Since g is homogeneous and $l_{12}(\deg(g)) = 0$ holds, we may further assume that $l_{12}(w_3) \le 0$ holds. The weights are arranged as follows, where w_3 lies somewhere on the opposite side of $H_{12} := \{x \in K_{\mathbb{Q}};\ l_{12}(x) = 0\}$ as w_ℓ.



This means that we have $Q(\gamma_{12\ell}) \subseteq Q(\gamma_{13\ell}) \cup Q(\gamma_{23\ell})$. Using Remark 3.2.4 and the circumstance that all faces of $\gamma_{13\ell}$ and $\gamma_{23\ell}$ are $\mathfrak{F}$-faces, we conclude that $u \in Q(\gamma_{13\ell})^\circ$ or $u \in Q(\gamma_{23\ell})^\circ$ holds, i.e. $\gamma_{13\ell}$ or $\gamma_{23\ell}$ is a relevant face. □

Lemma 3.5.4. *Let X be as in Setting 3.5.2 and assume that X is $\mathbb{Q}$-factorial. If $u \in Q(\gamma_{12})^\circ$ and $q \geq 6$ hold, then we have $\gamma_{ijk} \in \mathrm{rlv}(u)$ for some $i \in \{1,2\}$, $j \in \{3,4\}$, $k \in \{5,6\}$.*

Proof. We denote by $l_{12} \in \mathrm{Hom}(\mathrm{Cl}(X), \mathbb{Q})$ a linear form with $l_{12}(w_1) = 0 = l_{12}(w_2)$. Since g is homogeneous and $l_{12}(\deg(g)) = 0$ holds, renumbering of variables yields $l_{12}(w_3) \leq 0$ and $l_{12}(w_5) \geq 0$, i.e. w_3 and w_5 lie on opposite sides of the hypersurface cut out by l_{12}. Hence we have $Q(\gamma_{12}) \subseteq Q(\gamma_{135}) \cup Q(\gamma_{235})$. Using Remark 3.2.4 and the circumstance that all faces of γ_{135} and γ_{235} are $\mathfrak{F}$-faces, we conclude that $u \in Q(\gamma_{135})^\circ$ or $u \in Q(\gamma_{235})^\circ$ holds and thus γ_{135} or γ_{235} is a relevant face. □

Proposition 3.5.5. *Let X be as in Setting 3.5.2 and assume that X is locally factorial. Then the Picard group of X is torsion-free.*

Proof. According to Remark 3.2.5, it is sufficient to show that there is a three-dimensional relevant face. We distinguish the following two cases:

(1) g consists of squares,
(2) after renumbering of variables, we have $g = T_1T_2 + \ldots$.

Case (1): According to Carathéodory's theorem, there is an at most three-dimensional face τ of the positive orthant $\mathbb{Q}^{r+t}_{\geq 0}$ such that $u \in Q(\tau)^\circ$ holds. If τ is an $\mathfrak{F}$-face, then τ is a relevant face. Remark 3.2.4 shows that τ then is three-dimensional, which completes the proof in this situation. If τ is not an $\mathfrak{F}$-face, then, possibly after renumbering of variables, we have $\tau = \gamma_1$, $\tau = \gamma_{1r+1}$ or $\tau = \gamma_{1,r+1,r+2}$, where $u_1 = Q(e_{r+1})$ and $u_2 = Q(e_{r+2})$ denote the weights corresponding to the free variables S_1 and S_2. We show that only the third choice for τ is possible: If we had $u \in Q(\gamma_1)^\circ$ or $u \in Q(\gamma_{1r+1})^\circ$, then $\gamma_{12} \in \mathrm{rlv}(u)$ or $\gamma_{12r+1} \in \mathrm{rlv}(u)$ held, contradicting Remark 3.2.4. Thus we are in situation three, i.e. $\tau = \gamma_{1,r+1,r+2}$ holds. Since g is homogeneous, we have $w_1^0 = w_i^0$ for all $i = 1, \ldots, r$. Furthermore, $Q(\gamma_{i,j,r+1,r+2})^\circ = Q(\tau)^\circ$ holds for all $1 \leq i < j \leq r$, which shows that $\gamma_{i,j,r+1,r+2}$ is a relevant face for all $1 \leq i < j \leq r$. This yields $\mathrm{lin}_{\mathbb{Z}}(w_i, w_j, u_1, u_2) \geq \mathbb{Z}^3 \oplus \mathrm{Pic}(X)^{\mathrm{tor}}$ by Remark 3.2.5. In particular, we have $\mathrm{lin}_{\mathbb{Z}}(w_i^0, u_1^0, u_2^0) \geq \mathbb{Z}^3$ for all $1 \leq i < j \leq r$. Multiplying Q with an unimodular matrix from the left, we arrive at

$$(w_1, \ldots, w_r \,|\, u_1, u_2) \;=\; \left(\begin{array}{cccc|cc} 0 & 0 & \ldots & 0 & 1 & 0 \\ 0 & 0 & \ldots & 0 & 0 & 1 \\ 1 & 1 & \ldots & 1 & 0 & 0 \\ p & w_2^{\mathrm{tor}} & \ldots & w_r^{\mathrm{tor}} & p & p \end{array}\right),$$

where $p = 0_{\mathrm{Pic}(X)^{\mathrm{tor}}}$ holds. Since $\mathrm{lin}_{\mathbb{Z}}(w_1, w_2, u_1, u_2) = \mathrm{lin}_{\mathbb{Z}}(w_1, w_i, u_1, u_2)$ holds for all $2 \leq i \leq r$, we conclude $w_2^{\mathrm{tor}} = w_i^{\mathrm{tor}}$ for all $2 \leq i \leq r$. Applying again Remark 3.2.5, this time to $\gamma_{2,3,r+1,r+2}$, yields

$$\mathrm{lin}_{\mathbb{Z}}(w_2, u_1, u_2) \;=\; \mathrm{lin}_{\mathbb{Z}}(w_2, w_3, u_1, u_2) \;\geq\; \mathbb{Z}^3 \oplus \mathrm{Pic}(X)^{\mathrm{tor}},$$

which implies that $\mathrm{Pic}(X)$ is torsion-free.

Case (2): Here we have $\mathrm{Eff}(X) = Q(\sigma)$, where

$$\sigma \;:=\; \mathrm{cone}\big(e_i,\, e_{r+j};\;\; 1 \le i \le q,\; 1 \le j \le t\big)$$

holds. According to Carathéodory's theorem, there is an at most three-dimensional face τ of σ such that $u \in Q(\tau)^\circ$ holds. If τ is an $\mathfrak{F}$-face, then τ is a relevant face. Remark 3.2.4 shows that τ then is three-dimensional, which completes the proof in this situation. If τ is not an $\mathfrak{F}$-face, then, possibly after renumbering of variables, we are in one of the following subcases:

(2)(i) We have $g = T_1T_2 + T_3T_4 + \ldots$ and $\tau = \gamma_{123}$.
(2)(ii) We have $g = T_1T_2 + \ldots$ and $\tau = \gamma_{12r+1}$.
(2)(iii) We have $g = T_1T_2 + \ldots$ and $\tau = \gamma_{12}$.

Subcase (2)(i): If $q \ge 6$ holds, then Lemma 3.5.3 shows that there is a three-dimensional relevant face. Hence we only need to consider the situation that besides T_1T_2 and T_3T_4, the polynomial g consists of squares. But then $u \in \mathrm{Mov}(X)^\circ$ implies that $t \ge 1$ holds, i.e. there is some free variable T_{r+1}. Since we have

$$Q(\gamma_{123}) \;\subseteq\; Q(\gamma_{13r+1}) \;\cup\; Q(\gamma_{14r+1}) \;\cup\; Q(\gamma_{23r+1}) \;\cup\; Q(\gamma_{24r+1})\,,$$

and since all faces of the cones γ_{ijr+1}, $i \in \{1,2\}$, $j \in \{3,4\}$ are $\mathfrak{F}$-faces, Remark 3.2.4 shows that one of these cones is a relevant face. This completes the proof in Subcase (2)(i).
Subcase (2)(ii): Here we have $g = T_1T_2 + \ldots$ and $\tau = \gamma_{12r+1}$, where $u_1 = Q(e_{r+1})$ is the degree of the free variable S_1. If $q \ge 4$ holds, then Lemma 3.5.3 yields a three-dimensional relevant face. Hence we only need to consider the situation that besides T_1T_2, g consists of squares. In this situation, $\gamma_{123r+1} \in \mathrm{rlv}(u)$ holds. Note that we have $2w_3 = \deg(g) = w_1 + w_2$. Thus, Remark 3.2.5 applied to γ_{123r+1} yields

$$\mathrm{lin}_{\mathbb{Z}}(w_1, w_3, u_1) \;=\; \mathrm{lin}_{\mathbb{Z}}(w_1, w_2, w_3, u_1) \;\ge\; \mathrm{Pic}(X)\,,$$

which shows that $\mathrm{Pic}(X) \cong \mathbb{Z}^3$ holds.
Subcase (2)(iii): Here we have $g = T_1T_2 + \ldots$ and $\tau = \gamma_{12}$. In case $q \ge 6$ holds, Lemma 3.5.4 completes the proof. Now we consider the case $q \le 4$. If $q = 2$ held, then T_3^2 would be a square of g and thus γ_{123} would be a relevant face. Since $Q(\gamma_{123})$ is at most two-dimensional, this contradicts Remark 3.2.4. Hence we are in the case $q = 4$. Since $\mathrm{Mov}(X)$ is of full dimension, there is a free variable S_1. Note that we have

$$Q(\gamma_{12}) \;\subseteq\; Q(\gamma_{13r+1}) \;\cup\; Q(\gamma_{14r+1}) \;\cup\; Q(\gamma_{23r+1}) \;\cup\; Q(\gamma_{24r+1})\,.$$

Since all faces of the cones γ_{ijr+1}, $i \in \{1,2\}$, $j \in \{3,4\}$ are $\mathfrak{F}$-faces, Remark 3.2.4 shows that one of these cones is a relevant face, i.e. $\mathrm{Pic}(X) \cong \mathbb{Z}^3$ holds. $\square$

Problem 3.5.6. Generalize Proposition 3.5.5 to higher Picard numbers or give an example of a locally factorial intrinsic quadric with torsion in $\mathrm{Cl}(X)$.

Lemma 3.5.7. *Let X be as in Setting 3.5.2. If there are odd pairwise different integers $1 \le a, b, c \le q-1$ such that $\tau_0 := \gamma_{abc}$ and $\tau_1 := \gamma_{a+1,b+1,c+1}$ are relevant faces, then X is not locally factorial.*

Proof. Assume that X is locally factorial. Since there is a three-dimensional relevant face, Remark 1.3.3 implies that $\mathrm{Pic}(X) \cong \mathbb{Z}^3$ holds. Together with the homogeneity of the quadric, Remark 3.2.5 applied to τ_0 yields

$$(w_a, w_{a+1}, w_b, w_{b+1}, w_c, w_{c+1}) \;=\; \left(\begin{array}{cc|cc|cc} 1 & d_1 - 1 & 0 & d_1 & 0 & d_1 \\ 0 & d_2 & 1 & d_2 - 1 & 0 & d_2 \\ 0 & d_3 & 0 & d_3 & 1 & d_3 - 1 \end{array}\right),$$

where we denote by $d = (d_1, d_2, d_3)$ the degree of g. Note that

$$Q(\tau_0)^\circ \cap Q(\tau_1)^\circ \;\subseteq\; Q(\tau_{i,j})^\circ$$

holds for all $i, j \in \{a, b, c\}$, $i \neq j$ and $\tau_{i,j} := \mathrm{cone}(e_i, e_{i+1}, e_j, e_{j+1})$, i.e. all these cones $\tau_{i,j}$ are relevant faces. Applying Remark 3.2.5, we obtain $d = (1, 1, 1)$. This yields $\det(w_{a+1}, w_{b+1}, w_{c+1}) = 2$, contradicting Remark 3.2.5 applied to $\tau_1 \in \mathrm{rlv}(u)$. $\square$

Lemma 3.5.8. *Let X be as in Setting 3.5.2. If there are odd pairwise different integers $1 \leq a, b, c \leq q-1$ such that $\tau_0 := \gamma_{abc}$ and $\tau_1 := \gamma_{a,b,c+1}$ are relevant faces, then X is not locally factorial.*

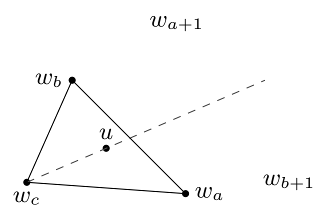
Proof. Assume that X is locally factorial. Since there is a three-dimensional relevant face, Remark 1.3.3 implies that $\mathrm{Pic}(X) \cong \mathbb{Z}^3$ holds. Together with the homogeneity of the quadric, Remark 3.2.5 applied to τ_0 and to τ_1 yields

$$(w_a, w_{a+1}, w_b, w_{b+1}, w_c, w_{c+1}) \; = \; \left(\begin{array}{cc|cc|cc} 1 & d_1 - 1 & 0 & d_1 & 0 & d_1 \\ 0 & d_2 & 1 & d_2 - 1 & 0 & d_2 \\ 0 & d_3 & 0 & d_3 & 1 & \pm 1 \end{array}\right),$$

where we denote by $d = (d_1, d_2, d_3)$ the degree of g. Note that the degree of the monomial $T_c T_{c+1}$ shows $d_3 \in \{0, 2\}$ $(*)$. Let $\tau_2 := \gamma_{a,a+1,b,b+1}$. Since $Q(\tau_0)^\circ \cap Q(\tau_1)^\circ$ is three-dimensional and contained in $Q(\tau_2)$, we conclude that the $\mathfrak{F}$-face τ_2 is a relevant face. Remark 3.2.5 applied to τ_2 yields $d_3 = \pm 1$, contradicting $(*)$. $\square$

Lemma 3.5.9. *Let X be as in Setting 3.5.2 and assume that $q < r$ holds, that is g contains a square. If there are odd integers $1 \leq a < b \leq q-1$ and an index $r + 1 \leq c \leq r + t$ such that $\tau_0 := \gamma_{abc}$ and $\tau_1 := \gamma_{a+1,b+1,c}$ are relevant faces, then X is not locally factorial.*

Proof. Assume that X is locally factorial. Since there is a three-dimensional relevant face, Remark 1.3.3 implies that $\mathrm{Pic}(X) \cong \mathbb{Z}^3$ holds. Possibly after renumbering of variables, the weights are arranged as follows,



where $\det(w_a, w_b, w_c) > 0$ holds. Together with the homogeneity of the quadric, Remark 3.2.5 applied to the relevant faces τ_0 and τ_1 yields

$$(w_a, w_{a+1} | w_b, w_{b+1} | w_r || w_c) \; = \; \left(\begin{array}{cc|cc|c||c} 1 & d_1 - 1 & 0 & d_1 & d_1/2 & 0 \\ 0 & d_2 & 1 & d_2 - 1 & d_2/2 & 0 \\ 0 & d_3 & 0 & d_3 & d_3/2 & 1 \end{array}\right)$$

and $1 = \det(w_{a+1}, w_c, w_{b+1}) = d_1 + d_2 - 1$ $(*)$, where we denote by $d = (d_1, d_2, d_3)$ the degree of g. Note that $Q(\tau_0)^\circ \cap Q(\tau_1)^\circ$ is three-dimensional and contained in $Q(\tau_i)$, $i = a, b$, where we set $\tau_i := \gamma_{i,i+1,r,c}$. We conclude that τ_a and τ_b are relevant faces. Together with Remark 3.2.5, this shows that $d_2 = \pm 2$ and $d_1 = \pm 2$ hold, contradicting $(*)$. $\square$

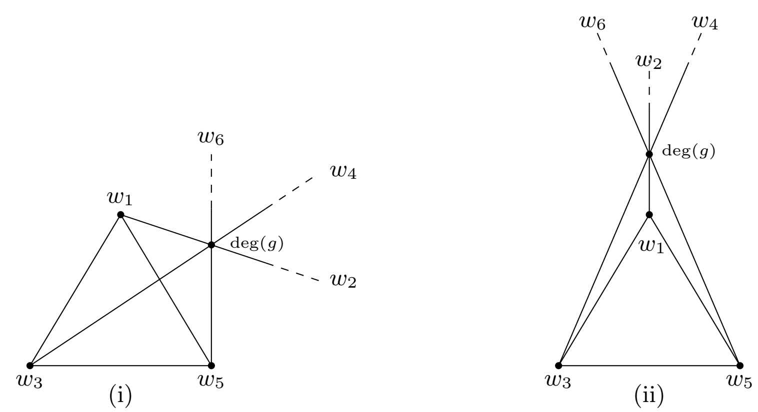
Lemma 3.5.10. *Let X be as in Setting 3.5.2 and assume that X is $\mathbb{Q}$-factorial and that $t = 0$ holds, i.e. X is a full intrinsic quadric. Then we have $q \geq 6$ and renumbering of variables yields $\gamma_{ijk} \in \mathrm{rlv}(u)$ for some $(i, j, k) \in \{1, 2\} \times \{3, 4\} \times \{5, 6\}$.*

Proof. Note that the moving cone of $\mathcal{R}(X)$ is of full dimension. This means that $q \geq 6$ holds, i.e. the quadric g is of the form $g = T_1T_2 + T_3T_4 + T_5T_6 + \ldots$. Note that the effective cone of X is given as $\mathrm{Eff}(X) = Q(\sigma)$, where we set $\sigma := \mathrm{cone}(e_i;\ 1 \leq i \leq q)$. According to Carathéodory's theorem, there is an at most three-dimensional face τ of σ such that $u \in Q(\tau)^\circ$ holds. After renumbering of variables, we have $\tau \preceq \gamma_{123}$ or $\tau \preceq \gamma_{135}$. Remark 3.2.4 together with the circumstance that all faces of type

γ_i, $1 \le i \le 6$, are $\mathfrak{F}$-faces, shows that τ is at least two-dimensional. If $\dim(\tau) = 2$ holds, then Remark 3.2.4 implies that τ is not an $\mathfrak{F}$-face, i.e. $\tau = \gamma_{12}$ holds. In this case, Lemma 3.5.4 completes the proof. If τ is three-dimensional, Lemma 3.5.3 completes the proof. $\square$

Lemma 3.5.11. *Let X be as in Setting 3.5.2 and assume that $q \ge 6$ holds. If γ_{135} is a relevant face and if X is locally factorial, then after renumbering of variables, $\deg(g) \in Q(\gamma_{135})$ and $\gamma_{135} \in \mathrm{rlv}(u)$ hold.*

Proof. If $\deg(g)$ is contained in $Q(\gamma_{135})$, there is nothing to show. Otherwise we are – after suitable renumbering of variables – in one of the following cases:



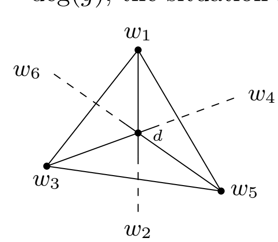
In Situation (i), Lemma 3.5.8 together with $\gamma_{135} \in \mathrm{rlv}(u)$ implies that γ_{136} and γ_{235} are not relevant. Note that we have

$$Q(\gamma_{135}) \subseteq Q(\gamma_{136}) \cup Q(\gamma_{236}) \cup Q(\gamma_{235})$$

and that all faces of $\gamma_{136}, \gamma_{236}$ and γ_{235} are $\mathfrak{F}$-faces. Since X is $\mathbb{Q}$-factorial, Remark 3.2.4 shows that γ_{236} is a relevant face. By exchanging T_1 and T_2 as well as T_5 and T_6, we arrive at $\deg(g) \in Q(\gamma_{135})$ and $\gamma_{135} \in \mathrm{rlv}(u)$. In Situation (ii), exchanging T_1 and T_2 yields the desired result. $\square$

Lemma 3.5.12. *Let X be as in Setting 3.5.2 and let $q \ge 6$. Assume that X is locally factorial and that γ_{135} is a relevant face with $\deg(g) \in Q(\gamma_{135})$. Then g contains no square.*

Proof. With $d = (d_1, d_2, d_3) := \deg(g)$, the situation is as follows:



Proposition 3.5.5 tells us that we have $\mathrm{Cl}(X) \cong \mathbb{Z}^3$. According to Remark 3.2.5, we may assume that

$$(w_1, \ldots, w_6) \;=\; \left(\begin{array}{cc|cc|cc} 1 & d_1 - 1 & 0 & d_1 & 0 & d_1 \\ 0 & d_2 & 1 & d_2 - 1 & 0 & d_2 \\ 0 & d_3 & 0 & d_3 & 1 & d_3 - 1 \end{array}\right)$$

holds. Since γ_{135} is a relevant face, u is contained in one of the cones

$$\mathrm{cone}(w_i, w_j, d) \setminus \mathrm{cone}(w_i, w_j), \quad i, j \in \{1, 3, 5\}, \quad i \neq j,$$

where Remark 3.2.4 shows that u is not contained in $\mathrm{cone}(w_i, w_j)$, $i, j \in \{1, 3, 5\}$. After renumbering of variables, we have $u \in \tau := \mathrm{cone}(w_1, w_3, d) \setminus \mathrm{cone}(w_1, w_3)$ and τ is three-dimensional. Since $Q(\gamma_{1234})^\circ$ contains τ°, we conclude that γ_{1234} is a relevant face. Remark 3.2.5 yields $d_3 = 1$, i.e. there is no square in g. □

Proposition 3.5.13. *A smooth full intrinsic quadric of Picard number three is isomorphic to an intrinsic quadric X with Cox ring*

$$\mathcal{R}(X) \;=\; \mathbb{K}[T_1, \ldots, T_r]/\langle T_1T_2 + \ldots + T_{r-1}T_r \rangle$$

where $r = \dim(X) + 4$ holds.

Proof. Let X be a smooth full intrinsic quadric of Picard number three. We may assume that X is as in Setting 3.5.2. According to Lemma 3.5.10 and Lemma 3.5.11, renumbering of variables yields $\mathcal{R}(X) = \mathbb{K}[T_1, \ldots, T_r]/\langle g \rangle$ with $g = T_1T_2 + T_3T_4 + T_5T_6 + \ldots$ as well as $\deg(g) \in Q(\gamma_{135})$, where γ_{135} is a relevant face. Lemma 3.5.12 shows that there is no square in g. This completes the proof. □

Problem 3.5.14. Generalize Proposition 3.5.13 to higher Picard numbers or give an example of a smooth full intrinsic quadric with relation $T_1T_2 + \ldots + T_r^2$.

Corollary 3.5.15. *Let X be a smooth full intrinsic quadric with $\varrho(X) = 3$. Then $\dim(X) \geq 4$ holds.*

Proof. According to Proposition 3.5.13, we may assume that the Cox ring of X is of the form

$$\mathcal{R}(X) \;=\; \mathbb{K}[T_1, \ldots, T_r]/\langle g \rangle \quad \text{with} \quad g = T_1T_2 + \ldots + T_{r-1}T_r$$

with $r = \dim(X) + 4$. Since the moving cone of $\mathcal{R}(X)$ is full dimensional, r is at least six. If $r = 6$ held, then $\mathrm{Mov}(X)$ would be contained in $\mathrm{cone}(w_1, w_3, w_5) \cap \mathrm{cone}(w_2, w_4, w_6)$, contradicting smoothness of X together with Lemma 3.5.7. Hence we obtain $r \geq 8$, i.e. $\dim(X) \geq 4$ holds. □

Lemma 3.5.16. *Let X be as in Setting 3.5.2 and set $r = 5$, $q = 4$, $t \geq 1$, i.e. we have $g = T_1T_2 + T_3T_4 + T_5^2$ and there is at least one free variable S_1. Consider an ample Weil divisor class $u \in \mathrm{Cl}(X)$. If X is locally factorial, then u is not contained in $Q(\gamma_{1234})$.*

Proof. Assume that u is contained in $Q(\gamma_{1234})$. Remark 3.2.4 together with the circumstance that all faces γ_{ij}, $i \in \{1, 2\}$, $j \in \{3, 4\}$ are $\mathfrak{F}$-faces, shows that $u \in Q(\gamma_{1234})^\circ$ holds. This means that we have $\gamma_{1234} \in \mathrm{rlv}(u)$. Moreover, note that

$$Q(\gamma_{1234})^\circ \;\subseteq\; Q(\gamma_{136}) \;\cup\; Q(\gamma_{326}) \;\cup\; Q(\gamma_{246}) \;\cup\; Q(\gamma_{416})$$

holds, where $u_1 = Q(e_6)$ denotes the degree of S_1. Possibly after renumbering of variables, we have $u \in Q(\gamma_{136})$. Remark 3.2.4 shows that γ_{136} is a relevant face. Since X is locally factorial, Remark 3.2.5 applied to γ_{136} yields

$$(w_1, \ldots, u_1) \;=\; \left(\begin{array}{cc|cc|c||c} 1 & d_1 - 1 & 0 & d_1 & d_1/2 & 0 \\ 0 & d_2 & 1 & d_2 & d_2/2 & 0 \\ 0 & d_3 & 0 & d_3 & d_3/2 & 1 \end{array}\right),$$

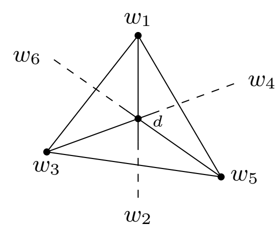
where $d = (d_1, d_2, d_3)$ denotes the degree of g. Applying Remark 3.2.5 to γ_{1234} then shows that $d_3 = \pm 1$ holds. But this contradicts $d_3/2 = w_5^3 \in \mathbb{Z}$, i.e. u is not contained in $Q(\gamma_{1234})$. □

Lemma 3.5.17. *Let X be as in Setting 3.5.2 and set $q \geq 6$. Consider an ample class $u \in \mathrm{Cl}(X)$. Assume that X is locally factorial and that u is contained in $Q(\gamma_{123456})$. Then suitable renumbering of variables yields*

$$(w_1, \ldots, w_6) \;=\; \left(\begin{array}{cc|cc|cc} 1 & d_1 - 1 & 0 & d_1 & 0 & d_1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{array}\right), \qquad d_1 \in \mathbb{Z}_{\geq 0}$$

as well as $\gamma_{135}, \gamma_{146}, \gamma_{1234}, \gamma_{1256} \in \mathrm{rlv}(u)$ *and* $u \in \mathrm{cone}(w_1, w_3, d) \cap Q(\gamma_{146})^\circ$, *where we denote by* $d = (d_1, d_2, d_3) := \deg(g)$ *the degree of* g.

Proof. According to Carathéodory's theorem, there is an at most three-dimensional face τ of $\mathrm{cone}(e_1, \ldots, e_6)$ such that $u \in Q(\tau)^\circ$ holds. After suitable renumbering of variables, we have $\tau \preceq \gamma_{135}$ or $\tau \preceq \gamma_{123}$. Remark 3.2.4 yields the three cases $\tau = \gamma_{12}$, $\tau = \gamma_{123}$ and $\tau = \gamma_{135}$. Lemma 3.5.3 and Lemma 3.5.4 show that suitable renumbering of variables yields $\gamma_{135} \in \mathrm{rlv}(u)$. According to Lemma 3.5.11, we may assume that $\deg(g) \in Q(\gamma_{135})$ holds.



Since γ_{135} is a relevant face, u is contained in one of the cones

$$\mathrm{cone}(w_i, w_j, d) \setminus \mathrm{cone}(w_i, w_j), \quad i, j \in \{1, 3, 5\}, \quad i \neq j,$$

where Remark 3.2.4 shows that u is not contained in $\mathrm{cone}(w_i, w_j)$, $\; i, j \in \{1, 3, 5\}$. After renumbering of variables, we have $u \in \tau := \mathrm{cone}(w_1, w_3, d) \setminus \mathrm{cone}(w_1, w_3)$ and τ is a three-dimensional cone. Since $Q(\gamma_{1234})^\circ$ contains τ°, we conclude that the cone γ_{1234} is a relevant face. Recall that Proposition 3.5.5 tells us that we have $\mathrm{Cl}(X) \cong \mathbb{Z}^3$. Remark 3.2.5 applied to γ_{135} and γ_{1234} yields

$$(w_1, \ldots, w_6) \;=\; \left(\begin{array}{cc|cc|cc} 1 & d_1 - 1 & 0 & d_1 & 0 & d_1 \\ 0 & d_2 & 1 & d_2 - 1 & 0 & d_2 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{array}\right).$$

Furthermore we have $d \in Q(\gamma_{135}) = \mathbb{Q}^3_{\geq 0}$, i.e. $d_1, d_2 \geq 0$ holds. Note that we have

$$\mathrm{cone}(w_1, w_3, d) \;\subseteq\; \mathrm{cone}(w_1, w_5, w_6) \cup \mathrm{cone}(w_3, w_5, w_6) \cup \mathrm{cone}(w_1, w_3, w_6).$$

Lemma 3.5.8 applied to γ_{135} implies that u is not contained in $\mathrm{cone}(w_1, w_3, w_6)^\circ$. After renumbering of variables, we may assume that u is contained in the three-dimensional cone generated by w_1, w_5 and w_6. Remark 3.2.4 together with the circumstance that γ_{15} and γ_{16} are $\mathfrak{F}$-faces, shows that u is not contained in $Q(\gamma_{15}) \cup Q(\gamma_{16})$. We obtain $u \in Q(\gamma_{1256})^\circ$, i.e. γ_{1256} is a relevant face. Thus, Remark 3.2.5 yields $d_2 = 1$. Applying Lemma 3.5.7 to $\gamma_{135} \in \mathrm{rlv}(u)$, we obtain that γ_{246} is not relevant. Remark 3.2.4 together with the circumstance that γ_{14}, γ_{16} and γ_{46} are $\mathfrak{F}$-faces, shows that $u \notin \mathrm{cone}(w_i, w_j)$ holds for all $i, j \in \{1, 4, 6\}$. Hence we have $u \in Q(\gamma_{146})^\circ \cap \mathrm{cone}(w_1, w_3, d)$. In particular, γ_{146} is a relevant face. $\square$

3.6. Proof of Theorem 3.3.5

In this section we give a proof of Theorem 3.3.5, i.e. of the classification of smooth intrinsic quadrics of Picard number three and dimension at most three.

Proof of Theorem 3.3.5. Let X be an at most three-dimensional smooth intrinsic quadric. Corollary 3.1.3 shows that we may assume that the Cox ring of X is given by $\mathcal{R}(X) = \mathbb{K}[T_1, \ldots, T_r, S_1, \ldots, S_t]/\langle g \rangle$ with $g = T_1T_2 + \ldots + T_{q-1}T_q + h$ for some $0 \leq q \leq r$ and some polynomial h given by

$$h \;=\; \begin{cases} T_{q+1}^2 + \ldots + T_r^2 & \text{if } q < r\,, \\ 0 & \text{if } q = r\,, \end{cases}$$

where we have $\deg(T_{q+k}) \neq \deg(T_{q+l})$ for all $1 \leq k < l \leq r - q$. According to Proposition 3.5.5, the Picard group of X is isomorphic to $\mathbb{Z}^3$. This means that the defining relation of $\mathcal{R}(X)$ contains at most one square. In a first step, we show that X is of dimension three. We then prove that all three-dimensional intrinsic quadrics that are smooth arise from the data sets in the table of Theorem 3.3.5. Note that on the other hand, all data sets listed in this table define smooth varieties by Lemma 3.1.6. In the very end of the proof, we prove the statement on (almost) Fano varieties.

First assume that X is of dimension one. This means that the Cox ring of X is given by $\mathcal{R}(X) = \mathbb{K}[T_1, \ldots, T_5]/\langle T_1T_2 + T_3T_4 + T_5^2\rangle$. But this is not possible since the moving cone of $\mathcal{R}(X)$ then is not full-dimensional.

Now we consider the case $\dim(X) = 2$. If the Cox ring of X was given by $\mathcal{R}(X) = \mathbb{K}[T_1, \ldots, T_6]/\langle T_1T_2 + T_3T_4 + T_5T_6\rangle$, then Lemma 3.5.7 would show that X is not smooth. Hence we have $\mathcal{R}(X) = \mathbb{K}[T_1, \ldots, T_6]/\langle T_1T_2 + T_3T_4 + T_5^2\rangle$. Denote by $u \in \mathrm{Cl}(X)$ an ample Weil divisor class. Since $u \in \mathrm{Mov}(X)^\circ$ holds, we obtain $u \in Q(\gamma_{1234})$, contradicting Lemma 3.5.16. Thus we showed that X is not two-dimensional.

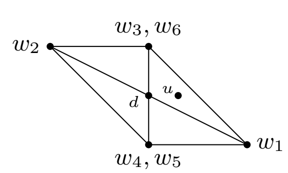
Hence X is three-dimensional and the Cox ring of X is given by $\mathcal{R}(X) = \mathbb{K}[T_1, \ldots, T_7]\langle g\rangle$, where we have $g = T_1T_2 + \ldots + T_5T_6 + T_7^2$, $g = T_1T_2 + \ldots + T_5T_6$ or $g = T_1T_2 + T_3T_4 + T_5^2$. By $u \in \mathrm{Cl}(X)$ we denote an ample Weil divisor class. If $g = T_1T_2 + \ldots + T_5T_6 + T_7^2$ held, we would have $u \in \mathrm{cone}(w_1, \ldots, w_6)^\circ$. Lemma 3.5.17 would show that after suitable renumbering of variables $\gamma_{135} \in \mathrm{rlv}(u)$ and $\deg(g) \in Q(\gamma_{135})$ held. But then Lemma 3.5.12 would yield that g contains no squares, a contradiction. Hence it remains to consider the two cases $g = T_1T_2 + \ldots + T_5T_6$ and $g = T_1T_2 + T_3T_4 + T_5^2$. By $w_1, \ldots, w_7$ we denote the degrees of the variables $T_1, \ldots, T_7$.

Case $\mathbf{g = T_1T_2 + \ldots + T_5T_6}$: We show that this yields varieties Nos. 1 and 2 in the table of Theorem 3.3.5.

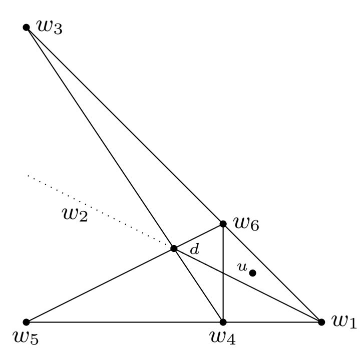
Here $u \in \mathrm{Mov}(X)^\circ$ yields $u \in Q(\gamma_{123456})$. Thus, Lemma 3.5.17 shows that suitable renumbering of variables yields $\gamma_{135}, \gamma_{1234}, \gamma_{1256}, \gamma_{146} \in \mathrm{rlv}(u)$ as well as $u \in \mathrm{cone}(w_1, w_3, d) \cap \mathrm{cone}(w_1, w_4, w_6)^\circ$ and

$$(w_1, \ldots, w_6) \;=\; \left(\begin{array}{cc|cc|cc} 1 & d_1 - 1 & 0 & d_1 & 0 & d_1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{array}\right), \qquad d_1 \geq 0\,.$$

We choose a hypersurface H intersecting the effective cone in its relative interior and illustrate the arrangement of weights in this two-dimensional picture. Note that we have $d = w_1 + w_2 = w_3 + w_4 = w_5 + w_6$. Moreover, if $d_1 = 0$ holds, then we have $w_3 = w_6$ and $w_4 = w_5$. If $d_1 \geq 1$ holds, then we have $w_4 \in \mathrm{cone}(w_1, w_5)$, $w_6 \in \mathrm{cone}(w_1, w_3)$ and $w_2 \in \mathrm{cone}(w_1, w_3, w_5)$. Depending on d_1, we give sketches of the different situations.



$d_1 = 0$



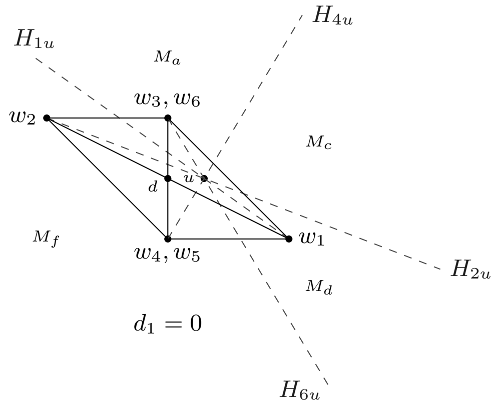
$d_1 > 0$

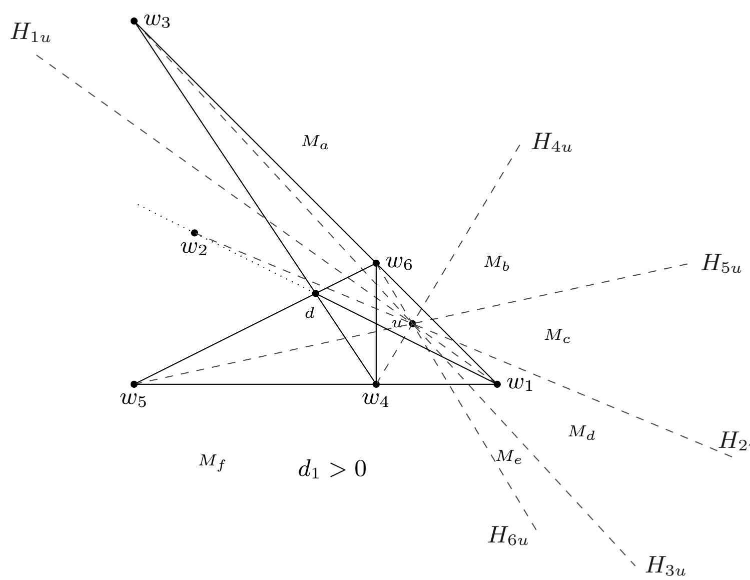
Let $0 \neq l_{iu} \in \mathrm{Hom}(K, \mathbb{Q}),\ i = 1, \dots, 6$, be linear forms such that

$$l_{iu}(w_i) = 0 = l_{iu}(u), \quad l_{iu}(w_1) > 0,\ i = 3, \dots, 6, \quad l_{2u}(w_4) > 0, \quad l_{1u}(w_3) > 0,$$

holds. Note that the faces γ_{i7} are $\mathfrak{F}$-faces for all $i = 1, \dots, 6$. Thus Remark 3.2.4 yields $l_{iu}(w_7) \neq 0$ for all $i = 1, \dots, 6$. As visualized below, there remain six possible places $M_a, \dots, M_f$ for w_7, where we set $H_{iu}^+ := \{x \in K_{\mathbb{Q}};\ l_{iu}(x) > 0\}$, $H_{iu}^- := \{x \in K_{\mathbb{Q}};\ l_{iu}(x) < 0\}$, $H_{iu} := \{x \in K_{\mathbb{Q}};\ l_{iu}(x) = 0\}$ as well as

$$\begin{array}{lll} M_a := H_{1u}^+ \cap H_{4u}^-, & M_b := H_{4u}^+ \cap H_{5u}^-, & M_c := H_{5u}^+ \cap H_{2u}^-, \\ M_d := H_{2u}^+ \cap H_{3u}^+, & M_e := H_{3u}^- \cap H_{6u}^+, & M_f := H_{6u}^- \cap H_{1u}^-. \end{array}$$





We show that w_7 is not contained in $M_b \cup M_e$. If w_7 was contained in M_b, then $\gamma_{175}, \gamma_{475}, \gamma_{247}, \gamma_{647}$ would be relevant faces. Applying Remark 3.2.5 to γ_{175} and to γ_{475} would yield $w_7^2 = 1$ and $d_1 = 1$. Thus, Remark 3.2.5 together with $\gamma_{247}, \gamma_{647} \in \mathrm{rlv}(u)$ would show that $w_7^1 - w_7^3 = 0$ and $w_7^1 - w_7^3 = 2$ hold, a contradiction. Similarily, if $w_7 \in M_e$ held, then $\gamma_{137}, \gamma_{637}, \gamma_{627}, \gamma_{647}$ would be relevant faces. Applying Remark 3.2.5 to γ_{137} and to γ_{637} would show that $w_7^3 = 1 = d_1$ holds. Thus, Remark 3.2.5 together with $\gamma_{627}, \gamma_{647} \in \mathrm{rlv}(u)$ would yield $w_7^1 - w_7^2 = 0$ and $w_7^1 - w_7^2 = 2$, a contradiction. Hence w_7 is contained in $M_a \cup M_c \cup M_d \cup M_f$. Note that we have

$$l_{6u}(w_1), l_{4u}(w_1) > 0, \qquad l_{6u}(w_i), l_{4u}(w_i) \leq 0,\ i = 2, \dots, 6.$$

Since u lies in the relative interior of the moving cone of $\mathcal{R}(X)$, we obtain that $l_{6u}(w_7) > 0$ and $l_{4u}(w_7) > 0$ hold. This means that w_7 is contained in $M_c \cup M_d$.

We first consider the case $w_7 \in M_c$. Here the covering collection of X is given by

$$\operatorname{cov}(u) \;=\; \{\gamma_{135}, \gamma_{164}, \gamma_{1234}, \gamma_{1256}, \gamma_{357}, \gamma_{257}, \gamma_{247}, \gamma_{647}\}\,.$$

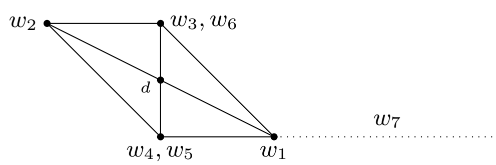
Remark 3.2.5 applied to γ_{357} yields $w_7^1 = 1$. The same remark together with the relevant faces $\gamma_{257}, \gamma_{247}$ and γ_{647} yields

$$0 \;=\; (d_1 - 1)w_7^2\,, \qquad 0 \;=\; d_1(w_7^2 - w_7^3)\,, \qquad 0 \;=\; d_1(w_7^2 + w_7^3)\,.$$

If $d_1 \neq 0$ held, we would obtain $w_7^2 = w_7^3 = 0$. Recall that u is contained in $\operatorname{cone}(w_1, w_3, d)$, which implies $u_2 \geq u_3$. But $u \in Q(\gamma_{257})^\circ$ together with $w_7 = (1,0,0)$ would yield $u_2 < u_3$, a contradiction. Hence we have $d_1 = 0$ and the first of the above equations shows $w_7^2 = 0$. Because of $u_2 \geq u_3$ and $u \in Q(\gamma_{257})^\circ$ we further obtain $w_7^3 < 0$. Thus, $Q = (w_1, \ldots, w_7)$ is as follows:

$$Q \;=\; \left(\begin{array}{cc|cc|cc||c} 1 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & w_7^3 \end{array}\right), \qquad w_7^3 < 0\,.$$

Since $w_7^2 = 0$ holds, w_7 lies on the hypersurface through w_1 and w_4. Because of $w_7^1 > 0$, the weights are arranged as follows, where w_7 lies somewhere on the dotted line:



Since $\operatorname{cone}(w_1, w_3, w_5) \subseteq \operatorname{cone}(w_3, w_5, w_7)$ holds, the semiample cone is the intersection of $\operatorname{cone}(w_1, w_3, w_5)$ and $\operatorname{cone}(w_2, w_5, w_7)$, which means that X is of type No. 1.

We now consider the case $w_7 \in M_d$. Here the covering collection of X is given by

$$\operatorname{cov}(u) \;=\; \{\gamma_{135}, \gamma_{164}, \gamma_{1234}, \gamma_{1256}, \gamma_{357}, \gamma_{273}, \gamma_{627}, \gamma_{647}\}\,.$$

Remark 3.2.5 applied to γ_{357} yields $w_7^1 = 1$. The same remark together with $\gamma_{273}, \gamma_{627}$ and γ_{647} yields

$$0 \;=\; (d_1 - 1)w_7^3\,, \qquad 0 \;=\; d_1(w_7^3 - w_7^2)\,, \qquad 0 \;=\; d_1(w_7^2 + w_7^3)\,.$$

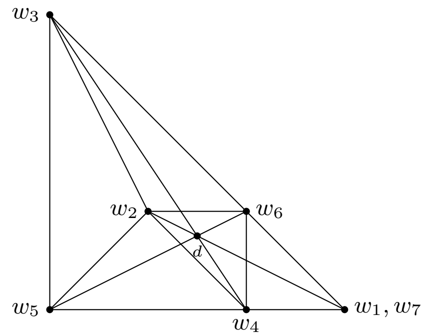
Thus we need to distinguish the subcases $d_1 = 0$ and $d_1 \neq 0$. In the first subcase the above relations show that $w_7^3 = 0$ holds. Exchanging the second and the third row of Q and renumbering the variables via $(3,4)(5,6)$ gives

$$Q \;=\; \left(\begin{array}{cc|cc|cc||c} 1 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & w_7^3 \end{array}\right),$$

$$\operatorname{cov}(u) \;=\; \{\gamma_{135}, \gamma_{164}, \gamma_{1234}, \gamma_{1256}, \gamma_{467}, \gamma_{274}, \gamma_{527}, \gamma_{537}\}\,.$$

We see that this coincides with the covering collection in the case $w_7 \in M_c$, which we treated above. In the second subcase the above relations show that $w_7^2 = w_7^3 = 0$ holds. Thus, $Q = (w_1, \ldots, w_7)$ and the arrangement of weights is as follows:

$$Q \;=\; \left(\begin{array}{cc|cc|cc||c} 1 & d_1 - 1 & 0 & d_1 & 0 & d_1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{array}\right), \quad d_1 > 0\,.$$



Note that we have $\mathrm{SAmple}(X) = \mathrm{cone}(w_1, w_4, w_6) \cap \mathrm{cone}(w_2, w_6, w_7)$, which shows that X is of type No. 2.

Case $\mathbf{g = T_1T_2 + T_3T_4 + T_5^2}$: We show that this yields varieties Nos. 3 – 6 in the table of Theorem 3.3.5.

Lemma 3.5.16 shows that $u \notin \tau := Q(\gamma_{1234})$ holds. Note that $u \in \mathrm{Mov}(X)^\circ$ yields $u \in \tau + \mathrm{cone}(w_i)$, $i = 6, 7$, i.e. we have $w_6, w_7 \notin \tau$. Moreover, we have

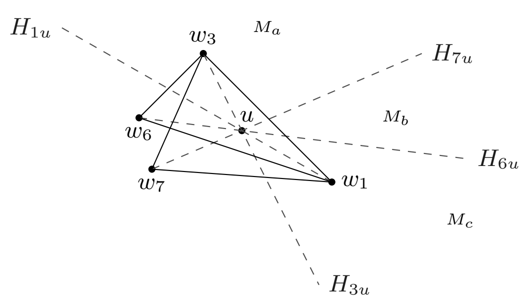
$$\tau + \mathrm{cone}(w_6) \;\subseteq\; Q(\gamma_{136}) \;\cup\; Q(\gamma_{326}) \;\cup\; Q(\gamma_{246}) \;\cup\; Q(\gamma_{416})\,.$$

Remark 3.2.4 shows that u is contained in the relative interior of one of the cones on the right-hand side. Thus, after renumbering of variables, γ_{136} is a relevant face. We distinguish the subcases $u \in Q(\gamma_{137})$ and $u \notin Q(\gamma_{137})$.

In the first subcase, Remark 3.2.4 shows that γ_{137} is a relevant face. Let $0 \neq l_{iu} \in \mathrm{Hom}(K, \mathbb{Q})$, $i = 1, 3, 6, 7$, be linear forms such that

$$l_{iu}(w_i) = 0 = l_{iu}(u)\,, \qquad l_{iu}(w_3) < 0\,,\ i = 6, 7\,, \qquad l_{iu}(w_6) < 0\,,\ i = 1, 3\,,$$

holds. After suitable renumbering of variables, the hypersurfaces $H_{iu} := \{x \in K_{\mathbb{Q}};\ l_{iu}(x) = 0\}$ are arranged as in the following picture and $\det(w_1, w_3, w_7)$ is strictly positive:



In the figures, M_a, M_b and M_c indicate the following sets of points:

$$\begin{aligned} M_a &= \{x \in K_{\mathbb{Q}};\ l_{1u}(x) > 0,\ l_{7u}(x) < 0\}\,, \\ M_b &= \{x \in K_{\mathbb{Q}};\ l_{6u}(x) < 0,\ l_{7u}(x) > 0\}\,, \\ M_c &= \{x \in K_{\mathbb{Q}};\ l_{3u}(x) > 0,\ l_{6u}(x) > 0\}\,. \end{aligned}$$

Note that the faces γ_{i6}, γ_{i7} are $\mathfrak{F}$-faces for all $i = 1, \ldots, 4$. Hence Remark 3.2.4 shows that $l_{6u}(w_i)$ and $l_{7u}(w_i)$, $i = 2, 4$, are non-zero. Together with $u \notin \tau$, this implies that w_2 and w_4 are contained in $M_a \cup M_b \cup M_c$. If $l_{6u}(w_2) > 0$ held, then the homogeneity of g would yield $l_{6u}(w_4) > 0$. But then we would have $l_{6u}(w_i) \geq 0$ for all $i \neq 3$, contradicting $u \in \mathrm{Mov}(X)^\circ$. Thus $l_{6u}(w_2) \leq 0$ holds. Lemma 3.5.9 applied to γ_{136} shows that this yields $l_{6u}(w_4) \leq 0$. The same Lemma shows that $l_{7u}(w_2)$ and $l_{7u}(w_4)$ are either strictly positive or strictly negative. But $l_{7u}(w_4)$ is not negative, since then homogeneity of g would yield $l_{7u}(w_i) \le 0$ for all $2 \le i \le 7$, contradicting $u \in \mathrm{Mov}(X)^\circ$. Thus we obtain $w_2, w_4 \in M_b$. We conclude that

$$\mathrm{cov}(u) \;=\; \{\gamma_{136}, \gamma_{137}, \gamma_{1256}, \gamma_{3457}, \gamma_{146}, \gamma_{237}, \gamma_{267}, \gamma_{467}\}$$

holds. Applying Remark 3.2.5 to γ_{136}, to γ_{146} and to γ_{137} yields

$$Q \;=\; \left(\begin{array}{cc|cc|c||cc} 1 & d_1-1 & 0 & d_1 & d_1/2 & 0 & w_7^1 \\ 0 & 2 & 1 & 1 & 1 & 0 & w_7^2 \\ 0 & d_3 & 0 & d_3 & d_3/2 & 1 & 1 \end{array}\right).$$

Furthermore, we have

$$\det(w_2, w_6, w_7) \;=\; (w_7^1 - d_1 w_7^2) + w_7^1 + w_7^2 \;=\; \det(w_4, w_6, w_7) + w_7^1 + w_7^2 .$$

Applying Remark 3.2.5 to γ_{267} and to γ_{467} shows that $w_7^1 = -w_7^2$ and $1 = w_7^1(1+d_1)$ hold. We conclude that either $d_1 = 0$, $w_7^1 = 1$ or $w_7^1 = -1$, $d_1 = -2$ holds.

We show that the latter is not possible: Assume that $w_7^1 = -1$, $d_1 = -2$ holds. Then Remark 3.2.5 applied to γ_{237} and to γ_{267} yields $d_3 = 4$ and $w_7^2 = 1$, which shows that the matrix Q is as follows:

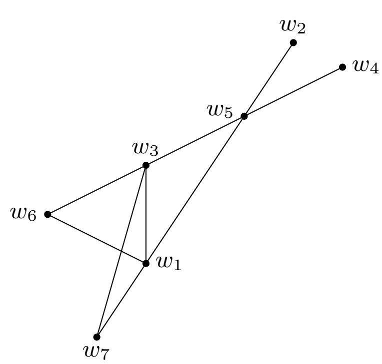
$$Q \;=\; \left(\begin{array}{cc|cc|c||cc} 1 & -3 & 0 & -2 & -1 & 0 & -1 \\ 0 & 2 & 1 & 1 & 1 & 0 & 1 \\ 0 & 4 & 0 & 4 & 2 & 1 & 1 \end{array}\right).$$

This shows that the intersection of the cones $Q(\gamma_{136})$ and $Q(\gamma_{137})$ is contained in $\mathrm{cone}(w_1, w_2, w_3)$. Thus we obtain the contradiction $u \in \tau$.

Hence we have $d_1 = 0$ and $w_7^1 = 1$. Remark 3.2.5 applied to γ_{237} yields $d_3 = -2$, which shows that the matrix Q is as follows:

$$Q \;=\; \left(\begin{array}{cc|cc|c||cc} 1 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & -1 \\ 0 & -2 & 0 & -2 & -1 & 1 & 1 \end{array}\right).$$

Note that we have $w_5 + w_6 = w_3$ as well as $w_5 + w_7 = w_1$. Thus the arrangement of weights is as follows:

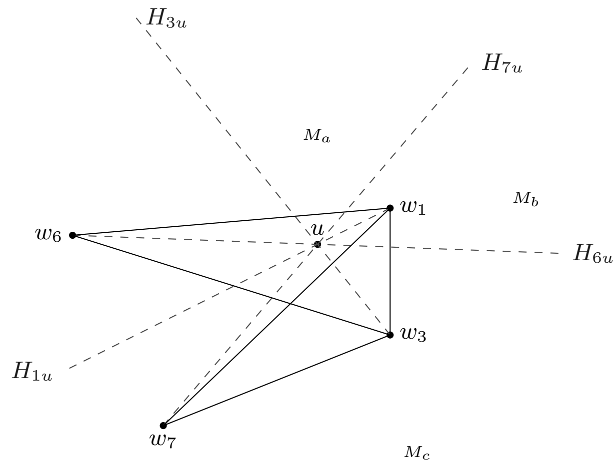


We conclude that $\mathrm{SAmple}(X) = Q(\gamma_{136}) \cap Q(\gamma_{137})$ holds, i.e. X is of type No. 6.

In the second subcase, we have $u \notin Q(\gamma_{137})$. Let $0 \ne l_{iu} \in \mathrm{Hom}(K, \mathbb{Q})$, $i = 1, 3, 6, 7$, be linear forms such that

$$l_{iu}(w_i) \;=\; 0 \;=\; l_{iu}(u), \qquad l_{iu}(w_3) \;<\; 0,\ i = 6, 7, \qquad l_{iu}(w_6) \;<\; 0,\ i = 1, 3,$$

holds and set $H_{iu} := \{x \in K_{\mathbb{Q}};\ l_{iu}(x) = 0\}$. After suitable renumbering of variables, the weights w_1, w_3, w_6 and w_7 are arranged as in the following picture and $\det(w_1, w_3, w_6)$ is strictly negative:



In the figures, M_a, M_b and M_c indicate the following sets of points:

$$M_a = \{x \in K_{\mathbb{Q}};\ l_{3u}(x) > 0,\ l_{7u}(x) > 0\},$$

$$M_b = \{x \in K_{\mathbb{Q}};\ l_{6u}(x) > 0,\ l_{7u}(x) < 0\},$$

$$M_c = \{x \in K_{\mathbb{Q}};\ l_{1u}(x) > 0,\ l_{6u}(x) < 0\}.$$

Since u is contained in the relative interior of the moving cone of $\mathcal{R}(X)$, $l_{7u}(w_2)$ or $l_{7u}(w_4)$ is strictly positive. In the first case, renumbering of variables via (12) would yield $u \in Q(\gamma_{136}) \cap Q(\gamma_{137})$. This is the subcase we treated above. Thus we now look at $l_{7u}(w_2) < 0$ and $w_4 \in M_a$. Lemma 3.5.9 applied to γ_{136} shows that $w_2 \notin M_c$ holds, i.e. we have $w_2 \in M_b$. Thus

$$\mathrm{cov}(u) = \{\gamma_{316}, \gamma_{326}, \gamma_{147}, \gamma_{247}, \gamma_{3456}, \gamma_{3457}, \gamma_{267}, \gamma_{167}\}$$

holds. Applying Remark 3.2.5 to γ_{147}, to γ_{247} and to γ_{167} yields

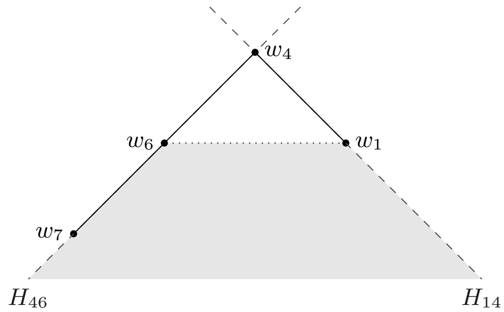
$$Q = \left(\begin{array}{cc|cc|c||cc} 1 & 1 & 2 & 0 & 1 & w_6^1 & 0 \\ 0 & d_2 & d_2-1 & 1 & d_2/2 & 1 & 0 \\ 0 & d_3 & d_3 & 0 & d_3/2 & w_6^3 & 1 \end{array}\right).$$

The same remark together with γ_{267} shows that $d_2 w_6^1 = 0$ holds. We distinguish the cases $w_6^1 = 0$ and $w_6^1 \neq 0$, $d_2 = 0$.

If $w_6^1 = 0$ holds, then Remark 3.2.5 applied to γ_{3456} and to γ_{316} yields $w_6^3 = 1$ and $d_2 = d_3$. Multiplying Q with an unimodular matrix from the left yields

$$Q = \left(\begin{array}{cc|cc|c||cc} 1 & 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & -1 \\ 0 & d_2 & d_2 & 0 & d_2/2 & 1 & 1 \end{array}\right).$$

Note that $w_4 + w_7 = w_6$ holds and that w_1, w_2, w_5 and w_3 lie on the same side of the hypersurface H_{46} through w_4 and w_6. Moreover, w_2 and w_5 lie on the hypersurface H_{16} through w_1 and w_6. If $d_2 \geq 0$ holds, then w_6, w_2, w_5 and w_3 lie on the same side of the hypersurface H_{14} through w_1 and w_4. Thus, in case $d_2 \geq 0$ holds, the weights are arranged as follows, where w_2 and w_5 lie somewhere on the dotted line and w_3 somewhere in the gray-shaded area:



Note that $Q(\gamma_{267}) \subseteq Q(\gamma_{247})$ holds, which shows that the semiample cone of X is given by the intersection of $Q(\gamma_{236})$ and $Q(\gamma_{267})$. Thus, X is of type No. 3. If $d_2 < 0$ holds, we multiply Q with an unimodular matrix from the left and obtain

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -d_2 & 0 & 1 \end{pmatrix} \cdot Q = \left(\begin{array}{cc|cc|c||cc} 1 & 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & -1 \\ -d_2 & 0 & -d_2 & 0 & -d_2/2 & 1 & 1 \end{array}\right).$$

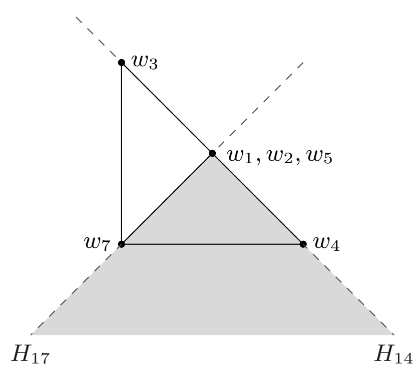
Renumbering the variables via $(1,2)$ then shows that X is of type No. 3.

If $w_6^1 \neq 0$, $d_2 = 0$ holds, Remark 3.2.5 applied to γ_{316} and to γ_{326} yields $w_6^3 = 1 - d_3$ as well as $0 = d_3(w_6^1 + 2)$. We distinguish the cases $d_3 = 0$ and $d_3 \neq 0$.

In case $d_3 = 0$ holds, the degree matrix is given by

$$Q = \left(\begin{array}{cc|cc|c||cc} 1 & 1 & 2 & 0 & 1 & w_6^1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right).$$

Note that w_6 lies on the same side of the hypersurface H_{14} through w_1 and w_4 as w_7 and on the same side of the hypersurface H_{17} through w_1 and w_7 as w_4. Thus the arrangement of weights is as follows, where w_6 lies somewhere in the gray-shaded area:

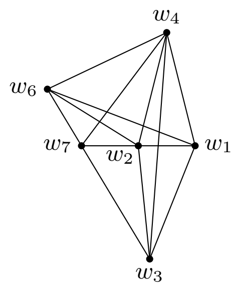


Note that $Q(\gamma_{147}) \cap Q(\gamma_{136})$ is contained in $Q(\gamma_{167})$, which shows that the semiample cone of X is given by the intersection of $Q(\gamma_{147})$ and $Q(\gamma_{136})$. Thus, X is of type No. 4.

In case $d_3 \neq 0, w_6^1 = -2$ hold, the degree matrix is given by

$$Q = \left(\begin{array}{cc|cc|c||cc} 1 & 1 & 2 & 0 & 1 & -2 & 0 \\ 0 & 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & d_3 & d_3 & 0 & d_3/2 & 1-d_3 & 1 \end{array}\right).$$

If $d_3 > 0$ holds, the arrangement of weights is as follows:



Note that the semiample cone of X is given by the intersection of $Q(\gamma_{247})$ and $Q(\gamma_{267})$. Thus, X is of type No. 5. If $d_3 < 0$ holds, we multiply Q with an unimodular matrix from the left and obtain

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -d_3 & 0 & 1 \end{pmatrix} \cdot Q \;=\; \left(\begin{array}{cc|cc|c||cc} 1 & 1 & 2 & 0 & 1 & -2 & 0 \\ 0 & 0 & -1 & 1 & 0 & 1 & 0 \\ -d_3 & 0 & -d_3 & 0 & -d_3/2 & d_3+1 & 1 \end{array}\right).$$

from the left. Renumbering the variables via $(1,2)$ then shows that X is of type No. 5.

To complete the proof, it remains to show the statement on Fano and truly almost Fano varieties. Recall that the anticanonical class of X is given by

$$-\mathcal{K}_X \;=\; \sum_{i=1}^{7} w_i - \deg(g) \;=\; \sum_{i=3}^{7} w_i\,.$$

In order to select the Fano and the truly almost Fano varieties among the smooth intrinsic quadrics of Picard number three and dimension three, it is enough to compute the anticanonical class of X via the above formula and to check in which cases $\mathcal{K}_X \in \mathrm{SAmple}(X)^\circ$ and $\mathcal{K}_X \in \mathrm{SAmple}(X) \setminus \mathrm{SAmple}(X)^\circ$ holds. Recall that the last column of the table of Theorem 3.3.5 contains three-dimensional cones of the form $\mathrm{cone}(x_i, x_j, x_k)$ whose intersection is the semiample cone of the respective variety. We denote for all pairwise different $i_1, i_2 \in \{i,j,k\}$ by $n_{i_1 i_2} \in \mathrm{Hom}(K, \mathbb{Q})$ a linear form satisfying $n_{i_1 i_2}(x_{i_1}) = n_{i_1 i_2}(x_{i_2}) = 0$ and $n_{i_1 i_2}(x_\ell) > 0$ for $\ell \in \{i,j,k\} \setminus \{i_1, i_2\}$. Note that a variety X of Theorem 3.3.5 is Fano if and only if $n_{i_1 i_2}(\mathcal{K}_X) > 0$ holds for all pairwise different $i_1, i_2 \in \{i,j,k\}$ of all cones $\mathrm{cone}(x_i, x_j, x_k)$ listed in the last column of Theorem 3.3.5 in the respective row. Similarily, a variety X of Theorem 3.3.5 is truly almost Fano if and only if $n_{i_1 i_2}(\mathcal{K}_X) \ge 0$ holds for all pairwise different $i_1, i_2 \in \{i,j,k\}$ of all cones $\mathrm{cone}(x_i, x_j, x_k)$ listed in the last column of Theorem 3.3.5 in the respective row, with equality for at least one $n_{i_1 i_2}$.

If X is of type No. 1, then the anticanonical class is given by $-\mathcal{K}_X = (1, 2, 2+a)$. Here n_{13} and n_{27} show that X is Fano if $a = -1$ holds and that X is truly almost Fano if and only if $a = 0$ or $a = -2$ hold.

If X is of type No. 2, then the anticanonical class is given by $-\mathcal{K}_X = (2a + 1, 2, 2)$. Note that $n_{46}(-\mathcal{K}_X) \ge 0$ yields $a \le 1/2$, contradicting $a > 0$. Hence there is no choice for a such that X is almost Fano.

If X is of type No. 3, then we have $a \ge 0$ and the anticanonical class is given by $-\mathcal{K}_X = (3, -1, 3a/2 + 2)$. Here $n_{27}(-\mathcal{K}_X) \ge 0$ yields $a \le 2/3$. Thus X is never truly almost Fano. Furthermore, X is Fano if and only if $a = 0$ holds.

If X is of type No. 4, then the anticanonical class is given by $-\mathcal{K}_X = (3+a, 1, 2)$. Here n_{36} and n_{47} show that X is Fano if $-2 \le a \le 0$ holds and that X is truly almost Fano if and only if $a = -3$ or $a = 1$ hold.

If X is of type No. 5, then we have $a > 0$ and the anticanonical class is given by $-\mathcal{K}_X = (1, 1, 2 + a/2)$. Note that $n_{26}(-\mathcal{K}_X) \ge 0$ yields $a \le 2/3$. Thus X is never almost Fano.

If X is of type No. 6, then the anticanonical class is given by $-\mathcal{K}_X = (1,2,-1)$. Note that $n_{13}(-\mathcal{K}_X)$ is strictly negative, which shows that X is neither Fano nor truly almost Fano. □

3.7. Proof of Theorems 3.3.6, 3.3.8 and 3.3.10

We now turn to the proof of our classification results for smooth intrinsic quadrics of dimension four and Picard number three.

Proof of Theorem 3.3.6. Let X be a smooth intrinsic quadric of Picard number three and dimension four. According to Proposition 3.5.5, the Picard group of X is isomorphic to $\mathbb{Z}^3$. Corollary 3.1.3 shows that we may assume that X is a standard intrinsic quadric. Thus there remain the following four possibilities for the relation g of the Cox ring $\mathcal{R}(X) = \mathbb{K}[T_1, \ldots, T_8]/\langle g \rangle$:

$$\begin{array}{ll} T_1T_2 + T_3T_4 + T_5T_6 + T_7T_8, & T_1T_2 + T_3T_4 + T_5T_6 + T_7^2, \\ T_1T_2 + T_3T_4 + T_5T_6, & T_1T_2 + T_3T_4 + T_5^2. \end{array}$$

In the remaining part of the proof we go through these four cases and show that we always end up with a variety listed in the table of Theorem 3.3.6. In order to provide a structure that is easily traceable, we outsource these four cases to Corollary 3.8.1 and to Propositions 3.7.1, 3.9.1 and 3.10.1. Moreover, note that all data sets listed in the table of Theorem 3.3.6 define indeed a smooth intrinsic quadric by Lemma 3.1.6. □

Proposition 3.7.1. *Let X be a four-dimensional intrinsic quadric of Picard number three with Cox ring $\mathcal{R}(X) = \mathbb{K}[T_1, \ldots, T_8]/\langle g \rangle$, $g = T_1T_2+T_3T_4+T_5T_6+T_7^2$. Then X is not smooth.*

Proof. Assume that X is a smooth four-dimensional intrinsic quadric of Picard number three with Cox ring as above and let u be an ample Weil divisor class. Proposition 3.5.5 tells us that $\mathrm{Cl}(X) \cong \mathbb{Z}^3$ holds. Furthermore, the effective cone of X is given by $\mathrm{Eff}(X) = \mathrm{cone}(w_1, \ldots, w_6, w_8)$, where we denote by $w_1, \ldots, w_8$ the degrees of the variables $T_1, \ldots, T_8$. Since we have $u \in \mathrm{Mov}(X)^\circ$, we obtain $u \in \mathrm{cone}(w_1, \ldots, w_6)^\circ$. Lemma 3.5.17 shows in particular that we may assume that $\gamma_{135} \in \mathrm{rlv}(u)$ and $\deg(g) \in Q(\gamma_{135})$ hold. But then Lemma 3.5.12 shows that g contains no squares, a contradiction. □

Proof of Theorems 3.3.8 and 3.3.10. Note that all smooth intrinsic quadrics of Picard number three and dimension four as well as their semiample cones are listed in the table of Theorem 3.3.6. Furthermore, the anticanonical class of X is given by

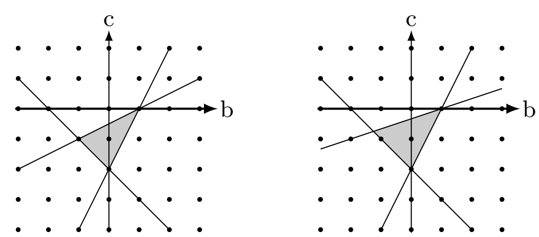
$$-\mathcal{K}_X \;=\; \sum_{i=1}^{8} w_i - \deg(g) \;=\; \sum_{i=3}^{8} w_i - \deg(g)\,.$$

In order to select the Fano and the truly almost Fano varieties among the varieties in the table of Theorem 3.3.6, it is enough to compute the anticanonical class of X via the above formula and to check in which cases $\mathcal{K}_X \in \mathrm{SAmple}(X)^\circ$ and $\mathcal{K}_X \in \mathrm{SAmple}(X) \setminus \mathrm{SAmple}(X)^\circ$ holds. Recall that the last column of the table of Theorem 3.3.6 contains three-dimensional cones of the form $\mathrm{cone}(x_i, x_j, x_k)$ whose intersection is the semiample cone of the respective variety. We denote for all pairwise different $i_1, i_2 \in \{i,j,k\}$ by $n_{i_1i_2} \in \mathrm{Hom}(K, \mathbb{Q})$ a linear form satisfying $n_{i_1i_2}(x_{i_1}) = n_{i_1i_2}(x_{i_2}) = 0$ and $n_{i_1i_2}(x_\ell) > 0$ for $\ell \in \{i,j,k\} \setminus \{i_1, i_2\}$. Note that a variety X of Theorem 3.3.6 is Fano if and only if $n_{i_1i_2}(\mathcal{K}_X) > 0$ holds for all pairwise different $i_1, i_2 \in \{i,j,k\}$ of all cones $\mathrm{cone}(x_i, x_j, x_k)$ listed in the last column of Theorem 3.3.6 in the respective row. Similarly, a variety X of Theorem 3.3.6 is truly almost Fano if and only if $n_{i_1i_2}(\mathcal{K}_X) \geq 0$ holds for all pairwise different

$i_1, i_2 \in \{i, j, k\}$ of all cones $\mathrm{cone}(x_i, x_j, x_k)$ listed in the last column of Theorem 3.3.6 in the respective row, with equality for at least one $n_{i_1 i_2}$.

If X is of type No. 1, then we have $-\mathcal{K}_X = (3a, 3, 3) = 3(w_4 + w_6) - 3aw_1$. In case a is strictly positive, the anticanonical class is contained in $\mathrm{cone}(-w_1, w_4+w_6)^\circ$ and X is neither Fano nor truly almost Fano. If $a = 0$ holds, then $-\mathcal{K}_X$ is contained in $\mathrm{cone}(w_4 + w_6)^\circ$, which shows that X is truly almost Fano.

If X is of type No. 2, then we have $-\mathcal{K}_X = (a + 1, 3, 2 + b + c)$. Note that $n_{35}(-\mathcal{K}_X) \geq 0$ and $n_{57}(-\mathcal{K}_X) \geq 0$ yield $-1 \leq a \leq 1/2$, i.e. X is Fano only if $a = 0$ holds. Furthermore, $n_{13}(-\mathcal{K}_X)$, $n_{17}(-\mathcal{K}_X) \geq 0$ and $n_{28}(-\mathcal{K}_X) \geq 0$ imply $c \geq -2 - b$, $c \geq 2b - 2$ and $0 \leq -ac + b - 3c - 1$. In the following picture, we illustrate the feasible region in the two cases $a = -1$ and $a = 0$:



$a = -1$ $a = 0$

We conclude that X is Fano if and only if $a = b = 0$ and $c = -1$ hold. Furthermore, X is truly almost Fano if and only if one of the following conditions is fulfilled:

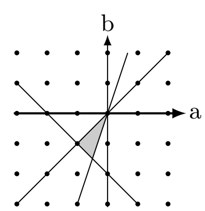
$$\begin{array}{ccccc} & -1 \leq a \leq 0,\ b = -1,\ c = -1 & \text{or} & -1 \leq a \leq 0,\ b = 0,\ c = -2 \\ \text{or} & -1 \leq a \leq 0,\ b = 1,\ c = 0 & \text{or} & a = -1,\ b = 0,\ c = -1. \end{array}$$

If X is of type No. 3, then we have $-\mathcal{K}_X = (1, 3+a, 2)$. Note that $n_{15}(-\mathcal{K}_X) \geq 0$ and $n_{28}(-\mathcal{K}_X) \geq 0$ yield $-3 \leq a \leq 1/2$. We obtain that X is Fano if and only if $-2 \leq a \leq 0$ holds and truly almost Fano if and only if $a = -3$ holds.

If X is of type No. 4, then we have $-\mathcal{K}_X = (1, 2, 2+a)$. Note that $n_{13}(-\mathcal{K}_X) \geq 0$ and $n_{78}(-\mathcal{K}_X) \geq 0$ yield $1 \geq a \geq -2$. We conclude that X is Fano if and only if $-1 \leq a \leq 0$ holds and truly almost Fano if and only if $a \in \{-2, 1\}$ holds.

If X is of type No. 5, then we have $-\mathcal{K}_X = (1, 1, 2)$. Thus, $n_{68}(-\mathcal{K}_X) < 0$ holds, which shows that X is neither Fano nor truly almost Fano.

If X is of type No. 6, then we have $-\mathcal{K}_X = (2, 2, 2 + a + b)$ and $0 > a \geq b$. Note that $n_{13}(-\mathcal{K}_X) \geq 0$ and $n_{27}(-\mathcal{K}_X) \geq 0$ yield $b \geq -2 - a$ and $b \geq 3a$. In the following picture, we illustrate the feasible region:



This shows that there is no choice for a and b such that X is Fano and because of $a, b < 0$, X is truly almost Fano if and only if $a = b = -1$ holds.

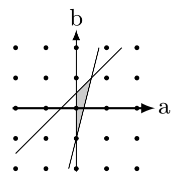
If X is of type No. 7, then we have $-\mathcal{K}_X = (2, 1, a + 2)$ and $a < 0$. Note that $n_{13}(-\mathcal{K}_X) \geq 0$ yields $0 \leq 2 + a$. We conclude that X is Fano if and only if $a = -1$ holds and truly almost Fano if and only if $a = -2$ holds.

If X is of type No. 8, then we have $-\mathcal{K}_X = (a+1, 3, 2)$. Note that $n_{35}(-\mathcal{K}_X) \geq 0$ and $n_{78}(-\mathcal{K}_X) \geq 0$ yield $0 \leq 1 + a$ as well as $0 \leq -2a$. We conclude that there is no choice for a such that X is Fano and X is truly almost Fano if and only if $-1 \leq a \leq 0$ holds.

If X is of type No. 9, then we have $a \geq 0$ and $-\mathcal{K}_X = (2a + b + 1, 3, 2)$. Note that $n_{46}(-\mathcal{K}_X) \geq 0$ and $n_{47}(-\mathcal{K}_X) \geq 0$ yield $0 \leq -3a + b + 1$ and $b \leq 1/2$. We conclude that X is Fano if and only if $a = 0 = b$ holds and truly almost Fano if and only if $a = 0, b = -1$ holds.

If X is of type No. 10, then we have $a > 0$ and $-\mathcal{K}_X = (2a+2, 2, 2)$. Note that $n_{27}(-\mathcal{K}_X)$ equals zero, i.e. there are no Fano varieties in this case. Furthermore $n_{46}(-\mathcal{K}_X) \geq 0$ yields $0 \leq -2a + 2$. Thus, X is truly almost Fano if and only if $a = 1$ holds.

If X is of type No. 11, then we have $a \geq 0$ and $-\mathcal{K}_X = (2a+b+1, c+2, 3)$. Note that $n_{18}(-\mathcal{K}_X)$, $n_{27}(-\mathcal{K}_X) \geq 0$ yields $0 \leq -2c + 2$ and $0 \leq c - 1$, i.e. there are no Fano varieties in this case and X is almost Fano only if $c = 1$ holds. Furthermore, $n_{68}(-\mathcal{K}_X) \geq 0$ and $n_{46}(-\mathcal{K}_X) \geq 0$ give $b \leq a + 1/2$ and $b \geq 4a - 1$. In the following picture, we illustrate the feasible region:



We conclude that X is truly almost Fano if and only if we have $a = 0$, $-1 \leq b \leq 0$ as well as $c = 1$.

If X is of type No. 12, then we have $-\mathcal{K}_X = (3, a+2, 0)$. Note that $n_{13}(-\mathcal{K}_X) = 0$ holds, i.e. there are no Fano varieties in this case. Furthermore, $n_{18}(-\mathcal{K}_X) \geq 0$ and $n_{27}(-\mathcal{K}_X) \geq 0$ show that X is truly almost Fano if and only if $a = -2 \leq a \leq -1$ holds.

If X is of type No. 13, then we have $-\mathcal{K}_X = (a + 2, b + 2, 2)$. Note that $n_{17}(-\mathcal{K}_X) \geq 0$ and $n_{18}(-\mathcal{K}_X) \geq 0$ give $-2 \leq b \leq 2$. Similarily, $n_{37}(-\mathcal{K}_X) \geq 0$ and $n_{38}(-\mathcal{K}_X) \geq 0$ give $-2 \leq a \leq 2$. We conclude that X is Fano if and only if $-1 \leq a, b \leq 1$ holds. Furthermore, X is truly almost Fano if and only if $a = \pm 2, -2 \leq b \leq 2$ or $b = \pm 2, -1 \leq a \leq 1$ holds.

If X is of type No. 14, then we have and $-\mathcal{K}_X = (2, 2, 2a + 2)$. Note that $n_{24}(-\mathcal{K}_X) \geq 0$ and $n_{13}(-\mathcal{K}_X) \geq 0$ give $-1 \leq a \leq 1$. Furthermore, looking at $n_{16}(-\mathcal{K}_X)$, $n_{25}(-\mathcal{K}_X)$, $n_{46}(-\mathcal{K}_X) \geq 0$ and $n_{35}(-\mathcal{K}_X) \geq 0$ yields $-1 \leq b \leq 1$ as well as $b \geq a - 1$ and $b \leq a + 1$. We conclude that X is Fano if and only if $a = b = 0$ holds, which is a subcase of No. 13. Furthermore, X is truly almost Fano if and only if $a = 1$, $0 \leq b \leq 1$, $a = 0$, $b = \pm 1$ or $a = -1$, $-1 \leq b \leq 0$ holds.

If X is of type No. 15, then we have and $-\mathcal{K}_X = (1, 3, -2)$. Since $n_{13}(-\mathcal{K}_X)$ is strictly negative, there are neither Fano nor truly almost Fano varieties in this case.

If X is of type No. 16, then we have $-\mathcal{K}_X = (4, 2a + 2, -1)$. Note that $n_{17}(-\mathcal{K}_X) \geq 0$ and $n_{27}(-\mathcal{K}_X) \geq 0$ yield $1/2 \geq a \geq -1/2$. We conclude that there is no choice for a for which X is almost Fano and that X is Fano if and only if $a = b = 0$ holds.

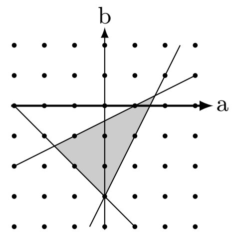
If X is of type No. 17, then we have $a \leq b \leq 0$ and $-\mathcal{K}_X = (4 + a, 2, 1)$. Note that $n_{37}(-\mathcal{K}_X) \geq 0$ and $n_{48}(-\mathcal{K}_X) \geq 0$ yield $a \geq -4$ and $a \leq 2$. We conclude that X is almost Fano if and only if $a = -4$ or $a = 2$ holds and that X is Fano if and only if $-3 \leq a \leq 1$ holds.

If X is of type No. 18, then $-\mathcal{K}_X = (2, a+3, 1)$ holds. Note that $n_{18}(-\mathcal{K}_X) \geq 0$ and $n_{28}(-\mathcal{K}_X) \geq 0$ yield $a/2 \leq 3/2$. We conclude that there is no choice for a and b for which X is almost Fano and that X is Fano only if $a = 1$ holds. Moreover, $n_{58}(-\mathcal{K}_X) \geq 0$ and $n_{68}(-\mathcal{K}_X) \geq 0$ show that $b = 0$ holds if X is Fano. In this case subtracting the first row of Q from the second shows that this variety is a subcase of No. 17.

If X is of type No. 19, then we have $a \leq b \leq 0$ and $-\mathcal{K}_X = (a + 1, b + 2, 2)$. Note that $n_{36}(-\mathcal{K}_X) \geq 0$ and $n_{16}(-\mathcal{K}_X) \geq 0$ give $a \geq -1$, $b \geq -2$. Moreover, $n_{36}(-\mathcal{K}_X) \geq 0$ and $n_{16}(-\mathcal{K}_X) \geq 0$ yield $a \leq 1$, $b \leq 2$. We conclude that X is

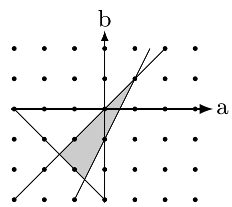
Fano if and only if $a = 0$ and $-1 \leq b \leq 1$ hold and truly almost Fano if and only if $a = \pm 1$, $-2 \leq b \leq 2$ or $a = 0, b = \pm 2$ hold.

If X is of type No. 20, then we have $-\mathcal{K}_X = (1, 3 + a + b, 3)$. Note that $n_{16}(-\mathcal{K}_X) \geq 0$, $n_{17}(-\mathcal{K}_X) \geq 0$ and $n_{28}(-\mathcal{K}_X) \geq 0$ yield $b \geq -3 - a$, $b \geq 2a - 3$ and $b \leq 1/2(a-1)$. In the following picture, we illustrate the feasible region:



This shows that X is Fano if and only if $a = 0$, $-2 \leq b \leq -1$ holds and truly almost Fano if and only if $(a, b) \in \{(\pm 1, -1), (-1, -2), (0, -3), (1, 0)\}$ holds.

If X is of type No. 21, then we have $a \geq b$ and $-\mathcal{K}_X = (2, 3+a+b, 3)$. Note that $n_{16}(-\mathcal{K}_X) \geq 0$ and $n_{27}(-\mathcal{K}_X) \geq 0$ yield $b \geq -3 - a$, $b \geq 2a - 1$. In the following picture, we illustrate the feasible region:



We conclude that X is Fano if and only if $a = b = 0$ or $a = b = -1$ holds and truly almost Fano if and only if $(a, b) \in \{(-1, -2), (0, -1), (1, 1)\}$ holds. If X is Fano, then we multiply Q with

$$\begin{pmatrix} -1 & 0 & 1 \\ 2 & 1 & -a-2 \\ 0 & 0 & 1 \end{pmatrix}$$

and renumber the variables via (12)(68). In this way, we see that the Fano varieties of Nos. 20 and 21 coincide.

If X is of type No. 22, then we have $-\mathcal{K}_X = (2, 2+a, 2)$ and $a \neq -2$. Note that $n_{37}(-\mathcal{K}_X) = 0$ holds, i.e. X is not Fano. Moreover, $n_{16}(-\mathcal{K}_X) \geq 0$ and $n_{27}(-\mathcal{K}_X) \geq 0$ give $-2 \leq a \leq 2$. We conclude that X is almost Fano if and only if $-1 \leq a \leq 2$ holds.

If X is of type No. 23, then we have $-\mathcal{K}_X = (a, 4, a+2)$. Note that $n_{27}(-\mathcal{K}_X) = 0$ holds, i.e. X is not Fano. Moreover, $n_{18}(-\mathcal{K}_X) \geq 0$ and $n_{36}(-\mathcal{K}_X) \geq 0$ give $0 \leq a \leq 2/3$, which yields $a = 0$. We conclude that X is almost Fano if and only if $a = 0$ holds.

If X is of type No. 24, then we have $-\mathcal{K}_X = (0, 4+a, 2)$. Note that $n_{36}(-\mathcal{K}_X) = 0$ holds, i.e. X is not Fano. Moreover, $n_{16}(-\mathcal{K}_X) \geq 0$ and $n_{27}(-\mathcal{K}_X) \geq 0$ show that X is almost Fano if and only $-4 \leq a \leq 0$ holds.

If X is of type No. 25, then we have $-\mathcal{K}_X = (a+1, 3, 2)$. Note that $n_{18}(-\mathcal{K}_X) < 0$ holds, i.e. X is not almost Fano.

If X is of type No. 26, then we have $a \in 2\mathbb{Z}_{\leq 0}$ and $-\mathcal{K}_X = (1, 2, 3a/2 + 2 + b)$. Note that $n_{38}(-\mathcal{K}_X) \geq 0$ yields $a \geq -4/3$. We conclude $a = 0$. Hence $n_{13}(-\mathcal{K}_X) \geq 0$ and $n_{28}(-\mathcal{K}_X) \geq 0$ give $b \geq -2$, $b \leq 2/3$. We conclude that X is Fano if and only if $a = 0$ and $-1 \leq b \leq 0$ hold and truly almost Fano if and only if $a = 0$ and $b = -2$ hold.

If X is of type No. 27, then we have $a \in 2\mathbb{Z}_{<0}$ and $-\mathcal{K}_X = (1, 1, a/2 + 3)$. Note that $n_{38}(-\mathcal{K}_X) \geq 0$ yields $a \geq -4/3$. We conclude that there is no choice for a such that X is almost Fano.

If X is of type Nos. 28 or 29, then we have $-\mathcal{K}_X = (-2, 3, 3a/2+3)$ and $-\mathcal{K}_X = (2, -1, 3-a/2)$, respectively. In the first case we have $n_{28}(-\mathcal{K}_X) < 0$ and in the latter $n_{16}(-\mathcal{K}_X) < 0$, which shows that X is neither Fano nor truly almost Fano.

If X is of type No. 30, then we have $a \in 2\mathbb{Z}_{\leq 0}$ and $-\mathcal{K}_X = (-1, 3, 3a/2+3)$. Note that $n_{36}(-\mathcal{K}_X) \geq 0$ yields $a \geq -4/3$. We conclude that X is Fano if and only if $a = 0$ holds and that there is no choice for a such that X is truly almost Fano. If X is Fano, we multiply Q with

$$\begin{pmatrix} -1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and renumber the variables via $(12)(68)$. In this way, we see that we are in the subcase No. 20, $a = -2$.

If X is of type No. 31, then we have $a \in 2\mathbb{Z}_{\leq 0}$ and $-\mathcal{K}_X = (2, 0, a/2+b+2)$. Note that $n_{16}(-\mathcal{K}_X) = 0$ holds, which shows that X is not Fano. Furthermore, $n_{37}(-\mathcal{K}_X) \geq 0$ and $n_{28}(-\mathcal{K}_X) \geq 0$ give $b \geq -5a/2$, $b \leq -a/2 + 2/3$. Together, this gives $a \geq -1/3$. We conclude that X is truly almost Fano if and only if $a = 0$ and $b = 0$ hold.

If X is of type No. 32, then we have $a \in 2\mathbb{Z}_{\leq 0}$ and $-\mathcal{K}_X = (-2, 4, 3a/2+b+2)$. Note that $n_{27}(-\mathcal{K}_X) = 0$ holds, which shows that X is not Fano. Furthermore, $n_{18}(-\mathcal{K}_X) \geq 0$ and $n_{36}(-\mathcal{K}_X) \geq 0$ give $b \leq a/2 + 2/3$, $b \geq -3a/2$. Together, this gives $a \geq -1/3$. We conclude that X is truly almost Fano if and only if $a = 0$ and $b = 0$ hold.

If X is of type No. 33, then we have $a \in 2\mathbb{Z}_{\leq 0}$ and $-\mathcal{K}_X = (0, 3, 3a/2+2)$. Note that $n_{36}(-\mathcal{K}_X) = 0$ holds, which shows that X is not Fano. Furthermore, $n_{13}(-\mathcal{K}_X) \geq 0$ yields $a \geq -4/3$. We conclude that a necessary condition for X being almost Fano is $a = 0$. Hence, $n_{18}(-\mathcal{K}_X) \geq 0$ and $n_{27}(-\mathcal{K}_X) \geq 0$ give $1/3 \leq b \leq 2/3$. Thus there are no almost Fano varieties in this case.

If X is of type Nos. 34 or 35, then we have $-\mathcal{K}_X = (1, 2, 0)$ and $-\mathcal{K}_X = (2, 1, 0)$, respectively. In both cases the anticanonical class is contained in $\operatorname{cone}(w_1, w_3)$ which is a face of the semiample cone of X. Thus, X is truly almost Fano.

If X is of type Nos. 36 or 37, then we have $-\mathcal{K}_X = (2, 1, -1)$ and $-\mathcal{K}_X = (0, 3, -1)$, respectively. In both cases $n_{13}(-\mathcal{K}_X)$ is strictly negative, which shows that X is neither Fano nor truly almost Fano. □

3.8. Smooth full intrinsic quadrics of Picard number three

In this section we prove Theorem 3.3.2 which provides the description of all smooth intrinsic quadrics of Picard number three that are full. Recall that *full* means that the Cox ring of X contains no free variable.

Proof of Theorem 3.3.2. We show that a smooth full intrinsic quadric X of Picard number three arises from Construction 3.3.1. According to Proposition 3.5.13, the Cox ring of X is of the form

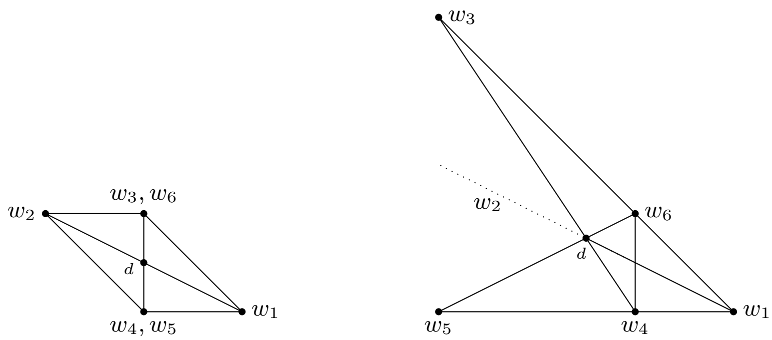
$$\mathcal{R}(X) \;=\; \mathbb{K}[T_1, \ldots, T_r]/\langle T_1T_2 + \ldots + T_{r-1}T_r \rangle$$

and Corollary 3.5.15 yields $r \geq 8$. Lemma 3.5.10 shows that we may assume that $\gamma_{135} \in \operatorname{rlv}(u)$ holds. Let $u \in \operatorname{Cl}(X)$ be an ample Weil divisor class. The homogeneity of g implies that u is contained in the three-dimensional cone $Q(\gamma_{123456})$. Hence Remark 3.2.4 shows that u is contained in $Q(\gamma_{123456})^\circ$. Lemma 3.5.17 shows that suitable renumbering of variables yields the relevant faces $\gamma_{135}, \gamma_{1234}, \gamma_{1256}, \gamma_{146}$

as well as $u \in \operatorname{cone}(w_1, w_3, d) \cap \operatorname{cone}(w_1, w_4, w_6)^\circ$ and

$$(w_1, \ldots, w_6) \;=\; \left(\begin{array}{cc|cc|cc} 1 & d_1 - 1 & 0 & d_1 & 0 & d_1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{array}\right), \qquad d_1 \geq 0\,, \qquad (*)$$

where $d = (d_1, 1, 1)$ denotes the degree of g. In order to illustrate the arrangement of weights, we choose a hypersurface H intersecting the effective cone in its relative interior and consider this two-dimensional picture. Note that we have $d = w_1 + w_2 = w_3 + w_4 = w_5 + w_6$. Moreover, if $d_1 = 0$ holds, then we have $w_3 = w_6$ and $w_4 = w_5$. If $d_1 \geq 1$ holds, then we have $w_4 \in \operatorname{cone}(w_1, w_5)$, $w_6 \in \operatorname{cone}(w_1, w_3)$ and $w_2 \in \operatorname{cone}(w_1, w_3, w_5)$. Depending on d_1, we give sketches of the different situations, where in the picture on the right-hand side, w_2 lies somewhere on the dotted line.



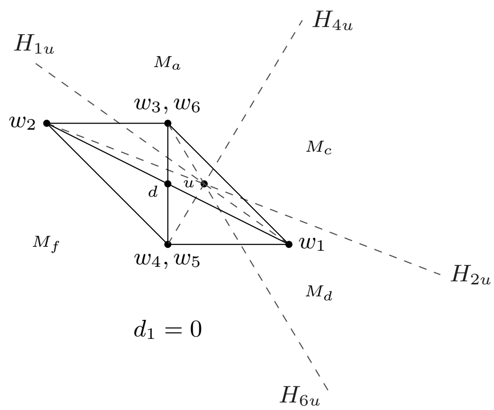
$d_1 = 0$ $\qquad\qquad$ $d_1 > 0$

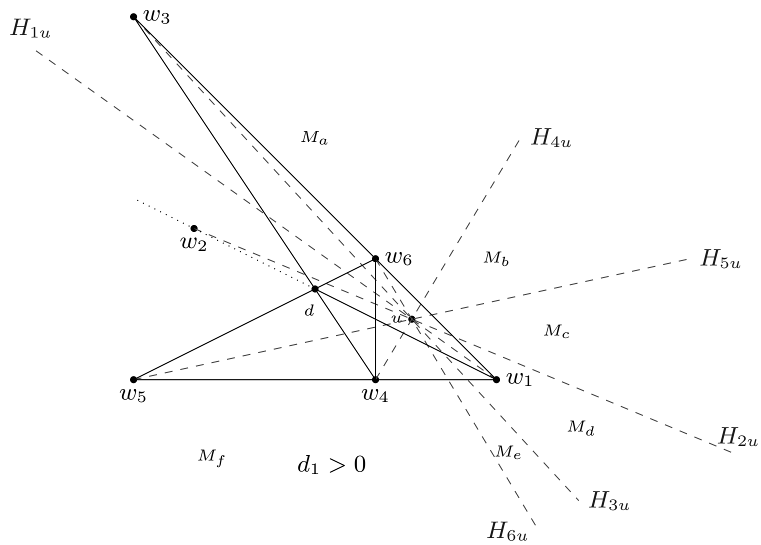
We now explain where further weights w_ℓ may lie. Let $0 \neq l_{iu} \in \operatorname{Hom}(K, \mathbb{Q})$, $i = 1, \ldots, 6$, be linear forms such that

$$l_{iu}(w_i) = 0 = l_{iu}(u)\,, \quad l_{iu}(w_1) > 0\,,\; i = 3, \ldots, 6\,, \quad l_{2u}(w_4) > 0\,, \quad l_{1u}(w_3) > 0\,,$$

hold. Note that the faces $\gamma_{i\ell}$ are $\mathfrak{F}$-faces for all $i = 1, \ldots, 6$, $\ell \geq 7$. Thus, Remark 3.2.4 yields $l_{iu}(w_\ell) \neq 0$ for all $i = 1, \ldots, 6$, $\ell \geq 7$. As visualized below, there remain six possible places $M_a, \ldots, M_f$ for w_ℓ, $\ell \geq 7$, where for all $i = 1, \ldots, 6$ we set $H_{iu}^+ := \{x \in K_\mathbb{Q};\; l_{iu}(x) > 0\}$, $H_{iu}^- := \{x \in K_\mathbb{Q};\; l_{iu}(x) < 0\}$, $H_{iu} := \{x \in K_\mathbb{Q};\; l_{iu}(x) = 0\}$ and

$$\begin{array}{lll} M_a := H_{1u}^+ \cap H_{4u}^-, & M_b := H_{4u}^+ \cap H_{5u}^-, & M_c := H_{5u}^+ \cap H_{2u}^-, \\ M_d := H_{2u}^+ \cap H_{3u}^+, & M_e := H_{3u}^- \cap H_{6u}^+, & M_f := H_{6u}^- \cap H_{1u}^-. \end{array}$$





As in the proof of Theorem 3.3.5, we see that further weights w_i, $i \geq 7$, are not contained in $M_b \cup M_e$. If $w_i, w_{i+1} \in M_f$ held for some odd $i \geq 7$ we would have γ_{13i}, $\gamma_{1,3,i+1} \in \mathrm{rlv}(u)$, contradicting Lemma 3.5.8. Hence renumbering of variables yields $w_i \in M_a \cup M_c \cup M_d$ for all odd $7 \leq i \leq r$. Choose an odd index $i \in \{7, \ldots, r\}$. Homogeneity of g shows that we are in one of the following cases

(i) $w_i \in M_a$, $w_{i+1} \in M_d \cup M_f$,
(ii) $w_i \in M_c$, $w_{i+1} \in M_f$,
(iii) $w_i \in M_d$.

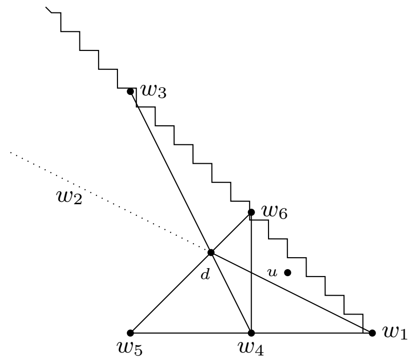
For an overview, we provide the following table, where elements of the covering collection are listed with respect to the different positions of w_i and w_{i+1}. As a matter of convenience, we list the indices a, b, c of the faces γ_{abc} in an order such that (w_a, w_b, w_c) is positively orientated, i.e. $\det(w_a, w_b, w_c) > 0$ holds.

Case	$\mathrm{cov}(u) \setminus \{\gamma_{135}, \gamma_{164}, \gamma_{1234}, \gamma_{1256}\}$
(i)	$\gamma_{1i4}, \gamma_{1,2,i,i+1}$
(ii)	$\gamma_{i35}, \gamma_{1,3,i+1}, \gamma_{i24}$
(iii)	$\gamma_{i35}, \gamma_{i32}, \gamma_{3,4,i,i+1}, \gamma_{i64}$

Case (i): $w_i \in M_a$, $w_{i+1} \in M_d \cup M_f$: Note that we will consider the case of a weight $w_\ell \in M_d$, $\ell \geq 7$, in Case (iii). Thus we may assume $w_{i+1} \in M_f$. In particular, we have $\gamma_{1,6,i+1} \in \mathrm{rlv}(u)$. Remark 3.2.5 together with the relevant face γ_{1i4} yields $w_i^2 = 1$. Applying the same remark to $\gamma_{1,2,i,i+1}$, we obtain $w_i^3 = 0$. Thus, we have

$$(w_1, w_2 | w_3, w_4 | w_5, w_6 | w_i, w_{i+1}) = \left(\begin{array}{cc|cc|cc|cc} 1 & d_1 - 1 & 0 & d_1 & 0 & d_1 & w_i^1 & d_1 - w_i^1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right).$$

The weights are arranged as follows, where w_2 lies somewhere on the dotted line and w_i somewhere on the zigzag line.



Note that SAmple(X) is contained in the intersection of the cones $Q(\gamma_{164})$, $Q(\gamma_{1i4})$ and $Q(\gamma_{1,6,i+1})$. We conclude that besides $w_1, \ldots, w_6, w_i, w_{i+1}$, there need to be further weights in order to ensure $u \in \mathrm{Mov}(X)^\circ$.

Case (ii): $w_i \in M_c$, $w_{i+1} \in M_f$: Remark 3.2.5 together with the relevant faces γ_{i35}, $\gamma_{1,3,i+1}$ and γ_{i24} yields $w_i^1 = 1$, $w_i^3 = 0$ and $w_i^2 = 0$. But this shows $w_1 = w_i$, i.e. we obtain $w_i \notin M_c$, a contradiction.

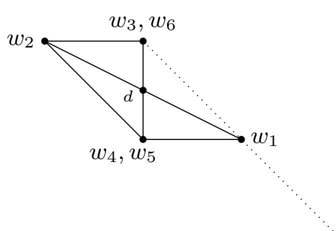
Case (iii): $w_i \in M_d$: Remark 3.2.5 together with $\gamma_{i35} \in \mathrm{rlv}(u)$ yields $w_i^1 = 1$. Applying again Remark 3.2.5, this time to γ_{i32} and to $\gamma_{3,4,i,i+1}$, yields

$$0 = w_i^3(1 - d_1) \quad \text{and} \quad 0 = d_1 w_i^3 .$$

This shows that $w_i^3 = 0$ holds. Now Remark 3.2.5 applied to γ_{i64} yields $d_1 = 0$ or $w_i^2 = 0$.

Case (iii.1): In the first subcase, i.e. if $d_1 = 0$ holds, we obtain the following arrangement of weights, where w_i lies somewhere on the dotted line:

$$(w_1, w_2 | w_3, w_4 | w_5, w_6 | w_i, w_{i+1}) \;=\; \left(\begin{array}{cc|cc|cc|cc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 & 0 & 1 & w_i^2 & 1 - w_i^2 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{array} \right).$$



Lemma 3.5.7 and $\gamma_{i32} \in \mathrm{rlv}(u)$ show that the $\mathfrak{F}$-face $\gamma_{i+1,4,1}$ is not a relevant face. Note that we have SAmple$(X) \subseteq \sigma := Q(\gamma_{i35}) \cap Q(\gamma_{i32}) \cap Q(\gamma_{135})$ and

$$\sigma \;\subseteq\; Q(\gamma_{i+1,4,1}) \;\cup\; Q(\gamma_{i+1,3,1}) .$$

Remark 3.2.4 together with $\gamma_{i+1,4,1} \notin \mathrm{rlv}(u)$ yields $\gamma_{i+1,3,1} \in \mathrm{rlv}(u)$. We conclude that the semiample cone of X is contained in

$$Q(\gamma_{135}) \;\cap\; Q(\gamma_{i35}) \;\cap\; Q(\gamma_{i32}) \;\cap\; Q(\gamma_{i+1,3,1}) .$$

Case (iii.2): In the second subcase we have $w_i^2 = 0$. Here we obtain the following:

$$(w_1, w_2 | w_3, w_4 | w_5, w_6 | w_i, w_{i+1}) \;=\; \left(\begin{array}{cc|cc|cc|cc} 1 & d_1 - 1 & 0 & d_1 & 0 & d_1 & 1 & d_1 - 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{array} \right).$$

Note that we have $w_1 = w_i$, $w_2 = w_{i+1}$ and SAmple(X) is a subset of the intersection of cone(w_1, w_4, w_6) and cone(w_1, w_2, w_6).

Now we discuss how the arrangement of the weights $w_1, \dots, w_6$ in $(*)$ can be enlarged by adding further variables w_i, w_{i+1}. The above reasoning shows that all smooth full intrinsic quadrics of Picard number three are obtained via combining $(*)$ with further monomials T_iT_{i+1}, where the weights $\deg(T_i) = w_i$ and $\deg(T_{i+1}) = w_{i+1}$ are as in (i), (iii.1) or (iii.2). Thus to complete the proof, it remains to combine $(*)$ with (i), (iii.1) and (iii.2) and to show that we arrive in the setting of Construction 3.3.1.

Note that combining (iii.1) with (iii.2) is the subcase of combining twice (iii.1) with $d_1 = 0$ and $w_i^2 = 0$. Moreover, we showed that in order to guarantee $u \in \mathrm{Mov}(X)^\circ$, $(*)$ together with one single monomial as in Case (i) is not possible alone. Thus we need to consider the combination of $(*)$ with at least two monomials of type (i), the combination of $(*)$ with monomials of type (iii.1), the combination of $(*)$ with monomials of type (iii.1) and (i), the combination of $(*)$ with monomials of type (iii.2), as well as the combination of $(*)$ with monomials of type (iii.2) and (i).

$(*)$ **and (i):** For $\ell \geq 6$, we have $w_\ell = (x_\ell, 1, 0)$ and $w_{\ell+1} = (d_1 - x_\ell, 0, 1)$. As argued above, there are at least two monomials of type (i). Assume that $\ell = 7$ is the index with $x_7 = \max(x_\ell)$ and $\ell = 9$ the index with $d_1 - x_9 = \max(d_1 - x_\ell)$. In order to ensure that u lies in the relative interior of the moving cone of X, we must have $x_7 > d_1$ and $d_1 - x_9 > d_1$, i.e. we have $w_7 \in \mathrm{cone}(w_1, w_6) \setminus \mathrm{cone}(w_6)$ and $w_{10} \in \mathrm{cone}(w_1, w_4) \setminus \mathrm{cone}(w_4)$. In addition, $u \in \mathrm{Mov}(X)^\circ$ shows $u \notin Q(\gamma_{1,7,10})$. Note that $\gamma_{1,4,7}$ and $\gamma_{1,6,10}$ are relevant faces. Thus, u lies in the cone

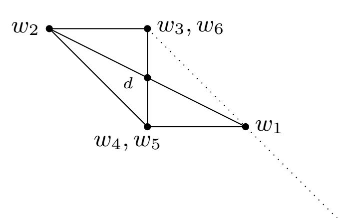
$$\tau \;:=\; Q(\gamma_{1,4,7}) \cap Q(\gamma_{1,6,10}).$$

Because of $\tau \subseteq Q(\gamma_{1,7,10}) \cap Q(\gamma_{2,7,10})$, we conclude that $\gamma_{2,7,10}$ is relevant. This gives $1 = \det(w_2, w_{10}, w_7) = d_1 - x_9 + x_7 + 1$. The above reasoning shows that $d_1 - x_9 + x_7 + 1 > 2d_1 + 1 \geq 1$ holds, a contradiction.

$(*)$ **and (iii.1):** Assume that w_7, w_8 and w_9, w_{10} are of type (iii.1). After renumbering of variables, we have $w_7^2 \geq w_9^2$ and

$$(w_1, w_2 | \dots | w_9, w_{10}) \;=\; \left(\begin{array}{cc|cc|cc|cc|cc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 1 & 1 & 0 & 0 & 1 & w_7^2 & 1 - w_7^2 & w_9^2 & 1 - w_9^2 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \end{array}\right).$$

Note that the weights are arranged as follows, where w_7 and w_9 lie on the dotted line



and where w_7 lies between w_3 and w_9. The semiample cone of X is contained in

$$\sigma \;:=\; Q(\gamma_{135}) \;\cap\; Q(\gamma_{735}) \;\cap\; Q(\gamma_{732}) \;\cap\; Q(\gamma_{1,3,10}).$$

Moreover, we have $\sigma \subseteq Q(\gamma_{389})$ and thus γ_{389} is a relevant face. Lemma 3.5.7 shows that $\gamma_{4,7,10} \notin \mathrm{rlv}(u)$ holds. Note that we have $\sigma \subseteq Q(\gamma_{4,7,10}) \cup Q(\gamma_{3,7,10})$. Remark 3.2.4 together with $\gamma_{4,7,10} \notin \mathrm{rlv}(u)$ yields $\gamma_{3,7,10} \in \mathrm{rlv}(u)$. We obtain

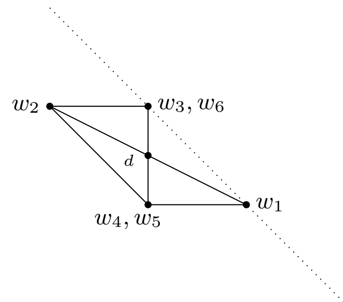
$$\mathrm{SAmple}(X) \;=\; \sigma \;\cap\; Q(\gamma_{3,7,10}).$$

Multiplying with an unimodular matrix from the left we thus arrive in the setting of Construction 3.3.1. We conclude that adding further monomials T_iT_{i+1}, $i \geq 11$, of type (iii.1) yields again a variety as in the setting of Construction 3.3.1.

$(*)$**, (iii.1) and (i):** Assume that the weights w_7, w_8 are of type (iii.1) and the weights w_9, w_{10} are of type (i). We obtain

$$(w_1, w_2 | \dots | w_9, w_{10}) \;=\; \left(\begin{array}{cc|cc|cc|cc|cc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & -1 & w_9^1 & -w_9^1 \\ 0 & 1 & 1 & 0 & 0 & 1 & w_7^2 & 1-w_7^2 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \end{array}\right).$$

Note that (iii.1) and (i) yield $\mathrm{SAmple}(X) \subseteq \sigma$, where σ is the intersection of $Q(\gamma_{357})$ and of $Q(\gamma_{149}) = Q(\gamma_{159})$. The weights are arranged as follows, where w_7 and w_9 lie on the dotted line:



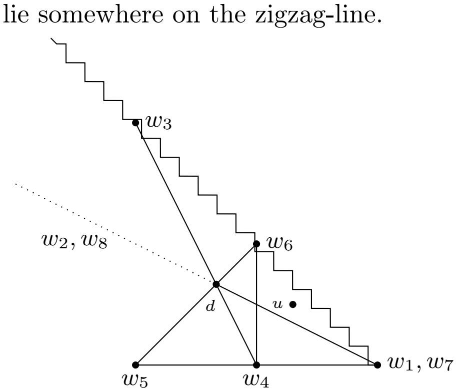
Since the semiample cone of X and thus σ is full-dimensional, we obtain $w_7^2 \le 0$ or $w_9^1 \le 0$. In the picture this means that w_7 lies below the hypersurface through w_1 and w_4 or that w_9 lies above the hypersurface through w_2 and w_3. Note that we have $\sigma \subseteq Q(\gamma_{579})$, which shows that γ_{579} is a relevant face. Thus, we may apply Remark 3.2.5 and obtain $1 = \det(w_5, w_7, w_9) = 1 - w_7^2 w_9^1$. This shows $w_7^2 = 0$ or $w_9^1 = 0$. If $w_7^2 = 0$ holds, then we are in a subcase of the combination (iii.2) and (i) with $d_1 = 0$, which we will treat below. If $w_9^1 = 0$ holds, then multiplying with an unimodular matrix from the left shows that we are in the setting of Construction 3.3.1.

$(*)$ **and (iii.2):** This is a subcase of the below discussed combination of $(*)$, (iii.2) and (i).

$(*)$**, (iii.2) and (i):** Assume that w_7, w_8 are of type (iii.2) and $w_9, w_{10}, w_{11}, w_{12}$ are of type (i). This means that the weights $(w_1, w_2 | \dots | w_{11}, w_{12})$ are given by

$$\left(\begin{array}{cc|cc|cc|cc|cc|cc} 1 & d_1-1 & 0 & d_1 & 0 & d_1 & 1 & d_1-1 & w_9^1 & d_1-w_9^1 & w_{11}^1 & d_1-w_{11}^1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right).$$

The weights are arranged as follows, where $w_2 = w_8$ lies somewhere on the dotted line and w_9 and w_{11} lie somewhere on the zigzag-line.



Possibly after renumbering of variables we arrive at $w_9^1 \ge w_{11}^1$. This means that w_9 lies in the cone generated by w_1 and w_{11}. Note that the semiample cone of X is contained in

$$\sigma \;:=\; Q(\gamma_{146}) \,\cap\, Q(\gamma_{168}) \,\cap\, Q(\gamma_{149}) \,\cap\, Q(\gamma_{1,6,12}) \cap\, Q(\gamma_{1,8,9})\,.$$

We show that $\gamma_{1,9,12}$ is a relevant face. We have $\sigma \subseteq Q(\gamma_{1,10,11})$ and thus $\gamma_{1,10,11} \in \mathrm{rlv}(u)$. Lemma 3.5.7 shows that $\gamma_{2,9,12} \notin \mathrm{rlv}(u)$ holds. Moreover, we have $\sigma \subseteq Q(\gamma_{1,9,12}) \cup Q(\gamma_{2,9,12})$. Remark 3.2.4 together with $\gamma_{2,9,12} \notin \mathrm{rlv}(u)$ yields $\gamma_{1,9,12} \in \mathrm{rlv}(u)$. We conclude

$$\mathrm{SAmple}(X) \;=\; \sigma \;\cap\; Q(\gamma_{1,9,12})\,.$$

Multiplying with an unimodular matrix from the left we arrive in the setting of Construction 3.3.1. We conclude that adding further monomials T_iT_{i+1}, $i \geq 11$, of type (iii.2) or type (i) yields again a variety as in Construction 3.3.1. $\square$

Corollary 3.8.1. *Let Y be a four-dimensional full intrinsic quadric of Picard number three. If Y is smooth, then Y is isomorphic to an intrinsic quadric X arising from the following data: We have $\mathrm{Cl}(X) = \mathbb{Z}^3$ and the Cox ring of X is given by $\mathcal{R}(X) = \mathbb{K}[T_1,\ldots,T_8]/\langle T_1T_2+T_3T_4+T_5T_6+T_7T_8\rangle$ with degree matrix*

$$(w_1,\ldots,w_8) \;=\; \left(\begin{array}{cc|cc|cc|cc} 1 & a-1 & 0 & a & 0 & a & 1 & a-1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right), \qquad a \in \mathbb{Z}_{\geq 0},$$

and semiample cone $\mathrm{SAmple}(X) = \mathrm{cone}(w_1, w_4+w_6, w_6)$.

Proof. The assertion follows immediately from the case $r = 8$ in Theorem 3.3.2: The degree matrix is given by

$$Q \;=\; \left(\begin{array}{cc|cc|cc|cc} 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 1 & 0 & 1 & 0 & 0 & 1 & a & 1-a \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right)$$

for some $a \in \mathbb{Z}_{\geq 0}$. Multiplying with the unimodular matrix

$$\begin{pmatrix} 0 & 1 & a-1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

from the left and suitably renumbering the variables yields the above form. $\square$

3.9. Proof of Proposition 3.9.1

In this section we give a description of all smooth four-dimensional intrinsic quadrics of Picard number three whose Cox ring contains two free variables.

Proposition 3.9.1. *Let X be a four-dimensional intrinsic quadric of Picard number three with Cox ring*

$$\mathcal{R}(X) \;=\; \mathbb{K}[T_1,\ldots,T_8]/\langle g\rangle, \qquad g \;=\; T_1T_2+T_3T_4+T_5T_6.$$

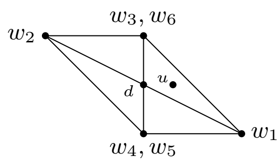
If X is smooth, then we have $\mathrm{Cl}(X) = \mathbb{Z}^3$ and X is isomorphic to one of the varieties 2 – 18 in the table of Theorem 3.3.6.

Proof. By u we denote an ample Weil divisor class and by $w_1,\ldots,w_8$ the degrees of the variables $T_1,\ldots,T_8$. With $\tau := \mathrm{cone}(w_i;\ i = 1,\ldots,6)$, we split the proof into the two parts $u \in \tau^\circ$ and $u \notin \tau^\circ$.

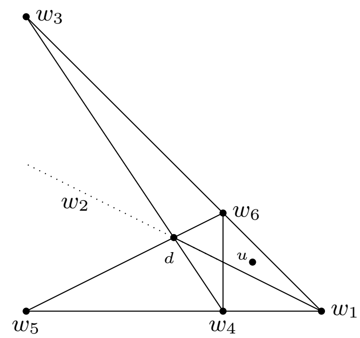
Part 1: First we consider the case $u \in \tau^\circ$. Lemma 3.5.17 shows that suitable renumbering of variables yields $\gamma_{135}, \gamma_{1234}, \gamma_{1256}, \gamma_{146} \in \mathrm{rlv}(u)$ as well as $u \in \mathrm{cone}(w_1,w_3,d) \cap \mathrm{cone}(w_1,w_4,w_6)^\circ$ and

$$(w_1,\ldots,w_6) \;=\; \left(\begin{array}{cc|cc|cc} 1 & d_1-1 & 0 & d_1 & 0 & d_1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{array}\right), \qquad d_1 \geq 0\,.$$

We choose a hypersurface H intersecting the effective cone in its relative interior and illustrate the arrangement of weights in this two-dimensional picture. Note that we have $d = w_1 + w_2 = w_3 + w_4 = w_5 + w_6$. Moreover, if $d_1 = 0$ holds, then we have $w_3 = w_6$ and $w_4 = w_5$. If $d_1 \geq 1$ holds, then we have $w_4 \in \operatorname{cone}(w_1, w_5)$, $w_6 \in \operatorname{cone}(w_1, w_3)$ and $w_2 \in \operatorname{cone}(w_1, w_3, w_5)$. Depending on d_1, we give sketches of the different situations.



$d_1 = 0$



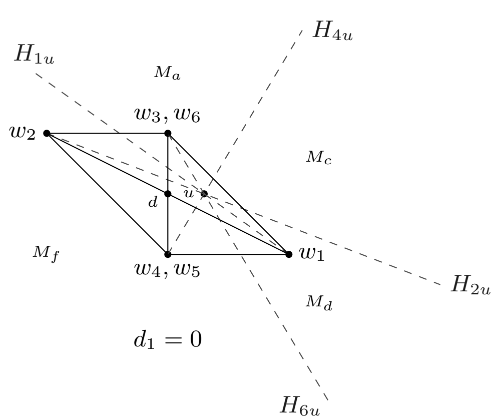
$d_1 > 0$

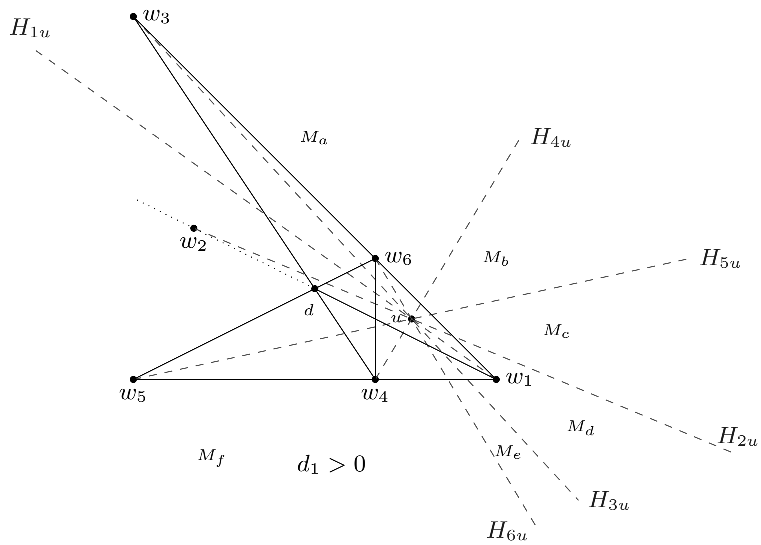
Let $0 \neq l_{iu} \in \operatorname{Hom}(K, \mathbb{Q}),\ i = 1, \ldots, 6$, be linear forms such that

$$l_{iu}(w_i) = 0 = l_{iu}(u)\,, \quad l_{iu}(w_1) > 0\,,\ i = 3, \ldots, 6\,, \quad l_{2u}(w_4) > 0\,, \quad l_{1u}(w_3) > 0\,,$$

holds. Note that the faces γ_{i7}, γ_{i8} are $\mathfrak{F}$-faces for all $i = 1, \ldots, 6$. Thus, Remark 3.2.4 yields $l_{iu}(w_7), l_{iu}(w_8) \neq 0$ for all $i = 1, \ldots, 6$. As visualized below, there remain six possible places $M_a, \ldots, M_f$ for w_7 and w_8, where for all $i = 1, \ldots, 6$ we set $H_{iu}^+ := \{x \in K_{\mathbb{Q}};\ l_{iu}(x) > 0\}$, $H_{iu}^- := \{x \in K_{\mathbb{Q}};\ l_{iu}(x) < 0\}$, $H_{iu} := \{x \in K_{\mathbb{Q}};\ l_{iu}(x) = 0\}$ and

$$\begin{array}{lll} M_a := H_{1u}^+ \cap H_{4u}^-, & M_b := H_{4u}^+ \cap H_{5u}^-, & M_c := H_{5u}^+ \cap H_{2u}^-, \\ M_d := H_{2u}^+ \cap H_{3u}^+, & M_e := H_{3u}^- \cap H_{6u}^+, & M_f := H_{6u}^- \cap H_{1u}^-. \end{array}$$





As in the proof of Theorem 3.3.5, we see that w_7 and w_8 are not contained in $M_b \cup M_e$. Hence w_7 and w_8 are contained in $M_a \cup M_c \cup M_d \cup M_f$. We now consider the remaining possibilities for w_7 and w_8. For an overview, we provide the following table, where elements of the covering collection are listed with respect to the different positions of w_7 and w_8. As a matter of convenience, we list the indices i, j, k of the faces γ_{ijk} in an order such that (w_i, w_j, w_k) is positively orientated, i.e. $\det(w_i, w_j, w_k) > 0$ holds. We denote by l_{7u} a linear form satisfying $l_{7u}(w_7) = l_{7u}(u) = 0$, $l_{7u}(w_3) < 0$. If $w_7 \in M_a$ holds, then l_{4u} and $u \in \mathrm{Mov}(X)^\circ$ show that $l_{4u}(w_8) > 0$ holds. In particular, w_8 is then contained in $M_c \cup M_d \cup M_f$. Furthermore, we have $l_{7u}(w_i) < 0$ for $2 \le i \le 6$, which shows that $l_{7u}(w_8) > 0$ holds. In particular if $w_7 \in M_a$ and $w_8 \in M_f$ hold, then we have $\gamma_{87i} \in \mathrm{rlv}(u)$ for $2 \le i \le 6$. If w_7 is contained in $M_c \cup M_d$ and $w_8 \in M_f$ holds, we distinguish the cases $l_{7u}(w_8) < 0$ and $l_{7u}(w_8) > 0$. Possibly after renumbering w_7 and w_8, we are left with the following cases:

w_7	w_8	$\mathrm{cov}(u) \setminus \{\gamma_{135}, \gamma_{164}, \gamma_{1234}, \gamma_{1256}\}$
(a)	(c)	$\gamma_{175}, \gamma_{174}, \gamma_{258}, \gamma_{358}, \gamma_{248}, \gamma_{648}, \gamma_{874}, \gamma_{875}$
(a)	(d)	$\gamma_{175}, \gamma_{174}, \gamma_{358}, \gamma_{283}, \gamma_{628}, \gamma_{872}, \gamma_{874}, \gamma_{875}, \gamma_{648}$
(a)	(f)	$\gamma_{175}, \gamma_{174}, \gamma_{138}, \gamma_{168}, \gamma_{872}, \gamma_{873}, \gamma_{874}, \gamma_{875}, \gamma_{876}$
(c)	(c)	$\gamma_{257}, \gamma_{357}, \gamma_{247}, \gamma_{647}, \gamma_{258}, \gamma_{358}, \gamma_{248}, \gamma_{648}$
(c)	(d)	$\gamma_{257}, \gamma_{357}, \gamma_{247}, \gamma_{647}, \gamma_{358}, \gamma_{283}, \gamma_{628}, \gamma_{648}, \gamma_{872}$
(c)	(f)	$\gamma_{257}, \gamma_{357}, \gamma_{247}, \gamma_{647}, \gamma_{138}, \gamma_{168}$; if $l_{7u}(w_8) > 0$: $\gamma_{872}, \gamma_{873}, \gamma_{876}$ if $l_{7u}(w_8) < 0$: $\gamma_{178}, \gamma_{478}, \gamma_{578}$
(d)	(d)	$\gamma_{357}, \gamma_{273}, \gamma_{627}, \gamma_{647}, \gamma_{358}, \gamma_{283}, \gamma_{628}, \gamma_{648}$
(d)	(f)	$\gamma_{357}, \gamma_{273}, \gamma_{627}, \gamma_{647}, \gamma_{138}, \gamma_{168}$; if $l_{7u}(w_8) > 0$: $\gamma_{873}, \gamma_{876}$ if $l_{7u}(w_8) < 0$: $\gamma_{278}, \gamma_{478}, \gamma_{578}$

We now apply Remark 3.2.5 to these cases and show that we end up with one of the varieties 2–11 in the table of Theorem 3.3.6. Note that Lemma 3.1.6 shows that the resulting varieties are smooth.

Case $\mathbf{w_7 \in M_a, w_8 \in M_c}$: We show that this leads to No. 2 in Theorem 3.3.6.

Applying Remark 3.2.5 to γ_{175}, γ_{358} and γ_{258} yields $w_7^2 = 1$, $w_8^1 = 1$ and $0 = w_8^2(d_1-1)$. The latter implies $w_8^2 = 0$ or $d_1 = 1$.

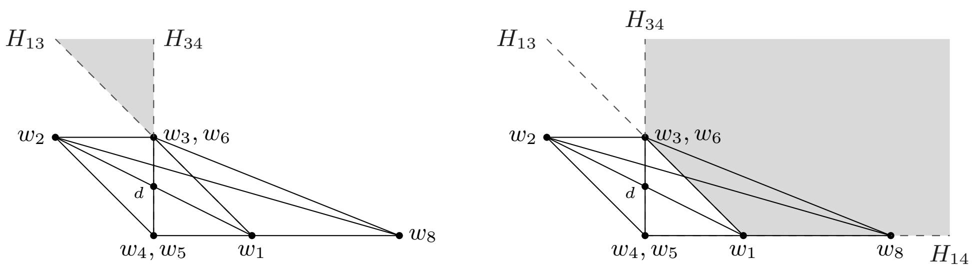
In the first case, Q is as follows:

$$Q \;=\; \left(\begin{array}{cc|cc|cc||cc} 1 & d_1-1 & 0 & d_1 & 0 & d_1 & w_7^1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & w_7^3 & w_8^3 \end{array}\right).$$

Remark 3.2.5 together with γ_{248} yields $d_1 w_8^3 = 0$, i.e. $d_1 = 0$ or $w_8^3 = 0$. Note that $u \in \operatorname{cone}(w_1, w_3, d)$ yields $u_2 \geq u_3$. Thus, $u \in Q(\gamma_{258})^\circ$ implies $w_8^3 < 0$. We conclude $d_1 = 0$ and $w_8^3 < 0$. Furthermore, $u \in Q(\gamma_{157})^\circ$ and $u_2 \geq u_3$ show that $w_7^3 \leq 0$ holds. Hence the degree matrix is of the form

$$Q \;=\; \left(\begin{array}{cc|cc|cc||cc} 1 & -1 & 0 & 0 & 0 & 0 & w_7^1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & w_7^3 & w_8^3 \end{array}\right), \qquad w_7^3 \leq 0,\ w_8^3 < 0\,.$$

Note that we have $w_1 \in \operatorname{cone}(w_5, w_8)$. We denote by H_{34}, H_{14} and H_{13} the hypersurfaces through w_3, w_4, through w_1, w_4 and through w_1, w_3, respectively. Since $w_7^2 > 0$ and $w_7^3 \leq 0$ holds, w_7 lies on the same side of H_{14} as w_3 and on the same side of H_{13} as w_8. We distinguish the situations $w_7^1 \leq 0$ and $w_7^1 > 0$. In both pictures, w_7 lies somewhere in the gray-shaded region:



$w_7^1 \leq 0$ $\qquad\qquad$ $w_7^1 > 0$

Since $w_3 = w_6, w_4 = w_5$ and $w_1 \in \operatorname{cone}(w_5, w_8)$ hold, the semiample cone equals

$$\operatorname{cone}(w_1, w_3, w_5) \,\cap\, \operatorname{cone}(w_1, w_7, w_5) \,\cap\, \operatorname{cone}(w_2, w_5, w_8) \,\cap\, \operatorname{cone}(w_5, w_7, w_8)\,,$$

i.e. X is of type No. 2.

Now we consider the case $d_1 = 1$. Applying Remark 3.2.5 to γ_{248} and to γ_{648} yields $w_8^2 - w_8^3 = 0$ and $w_8^2 + w_8^3 = 0$. We conclude that $w_8 = (1,0,0)$ holds. Note that $u \in \operatorname{cone}(w_1, w_3, d)$ implies $u_2 \geq u_3$, contradicting $u \in Q(\gamma_{248})^\circ$.

Case $\mathbf{w_7 \in M_a, w_8 \in M_d}$: We show that this leads to Nos. 3, 4 and 9 in Theorem 3.3.6.

Applying Remark 3.2.5 to γ_{175} and γ_{358} yields $w_7^2 = 1 = w_8^1$. Thus, $Q = (w_1, \ldots, w_8)$ is as follows:

$$Q \;=\; \left(\begin{array}{cc|cc|cc||cc} 1 & d_1-1 & 0 & d_1 & 0 & d_1 & w_7^1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & w_8^2 \\ 0 & 1 & 0 & 1 & 1 & 0 & w_7^3 & w_8^3 \end{array}\right), \qquad d_1 \geq 0\,.$$

Remark 3.2.5 applied to $\gamma_{283}, \gamma_{628}$ and γ_{648} yields

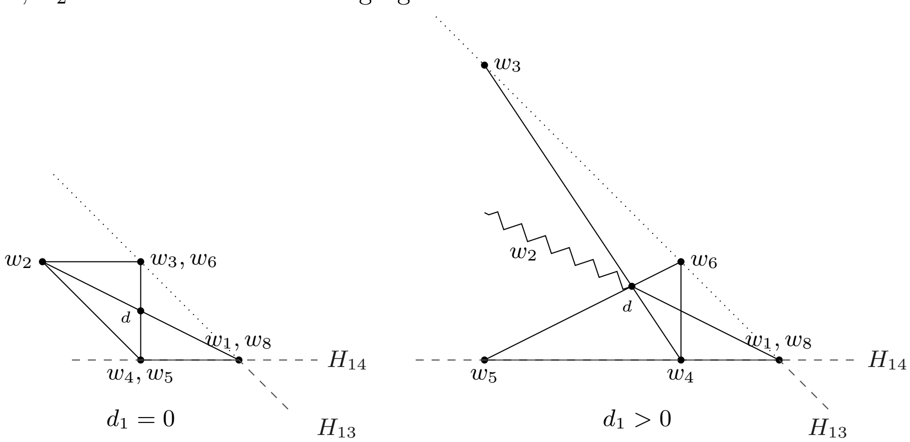
$$0 \;=\; (d_1-1)w_8^3\,, \quad 0 \;=\; d_1 w_8^3 - (d_1-1)w_8^3 - d_1 w_8^2\,, \quad 0 \;=\; d_1(w_8^2 + w_8^3)\,. \quad (*)$$

Inserting the first in the second and the second in the third equation yields $0 = d_1 w_8^2$. Together with the third equation, this gives the two cases $w_8^2 = w_8^3 = 0$ and $d_1 = 0$.

If $w_8^2 = w_8^3 = 0$ holds, Remark 3.2.5 applied to γ_{872} shows that $w_7^3 = 0$ holds. Thus the degree matrix is given as

$$Q \;=\; \left(\begin{array}{cc|cc|cc||cc} 1 & d_1-1 & 0 & d_1 & 0 & d_1 & w_7^1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right), \quad d_1 \geq 0\,.$$

We denote by H_{14} and H_{13} the hypersurfaces through w_1, w_4 and through w_1, w_3, respectively. Since $w_7^2 > 0$ and $w_7^3 = 0$ holds, w_7 lies on H_{13} and on the same side of H_{14} as w_3. In the situations $d_1 = 0$ and $d_1 \geq 0$, the weights are arranged as follows, where w_7 lies somewhere on the dotted line and in the picture on the right-hand side, w_2 lies somewhere on the zigzag line:



Note that we have $w_1 = w_8$, $w_4 \in \operatorname{cone}(w_1, w_5)$ and $w_6 \in \operatorname{cone}(w_1, w_3)$. Furthermore, we have $\operatorname{cone}(w_8, w_7, w_2) \subseteq \operatorname{cone}(w_8, w_7, w_4) \cap \operatorname{cone}(w_6, w_2, w_8)$. Thus, the semiample cone fulfills

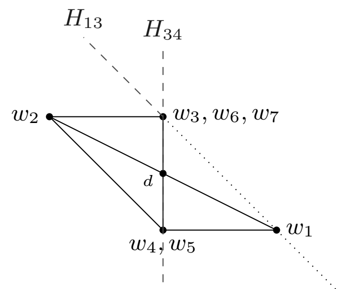
$$\operatorname{SAmple}(X) \;=\; \operatorname{cone}(w_1, w_4, w_6) \;\cap\; \operatorname{cone}(w_8, w_7, w_4) \;\cap\; \operatorname{cone}(w_6, w_2, w_8)\,,$$

i.e. X is of type No. 9.

We treat the case $d_1 = 0$. Here the first of the equations in $(*)$ shows that $w_8^3 = 0$ holds. Thus, Remark 3.2.5 applied to γ_{872} and to γ_{875} yields $0 = -w_7^3(w_8^2+1) - w_7^1 w_8^2$ and $0 = w_7^1 w_8^2$. If $w_8^2 = 0$ holds, then we are in a special case of the above treated case $w_8^2 = w_8^3 = 0$. If $w_7^1 = 0$ holds, then we have either $w_7^3 = 0$ or $w_8^2 = -1$. We first treat the subcase $w_7^3 = 0$. Here, the degree matrix is given by

$$Q \;=\; \left(\begin{array}{cc|cc|cc||cc} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & w_8^2 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right).$$

We denote by H_{13} and H_{34} the hypersurfaces through w_1, w_3 and through w_3, w_4, respectively. Since $w_8^1 > 0$ and $w_8^3 = 0$ hold, w_8 lies on H_{13} and on the same side of H_{34} as w_1. Thus, the weights are arranged as follows, where w_8 lies somewhere on the dotted line:



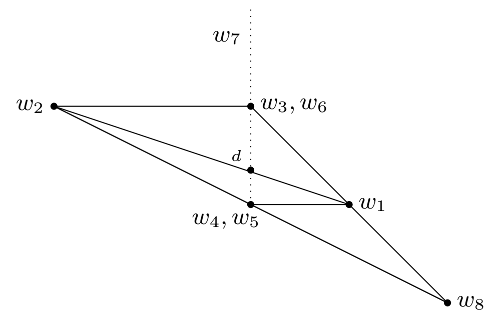
Because of $w_4 = w_5, w_3 = w_6 = w_7$ and

$$\operatorname{cone}(w_1, w_5, w_7) \ \cap \ \operatorname{cone}(w_2, w_3, w_8) \ \subseteq \ \operatorname{cone}(w_3, w_5, w_8)\,,$$

the semiample cone fulfills $\operatorname{SAmple}(X) = \operatorname{cone}(w_1, w_5, w_7) \cap \operatorname{cone}(w_2, w_3, w_8)$, i.e. X is of type No. 3. We now treat the subcase $w_8^2 = -1$. Here, the degree matrix is given by

$$Q \ = \ \left(\begin{array}{cc|cc|cc||cc} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 & 1 & 0 & w_7^3 & 0 \end{array}\right).$$

Since we have $w_1 \in \operatorname{cone}(w_3, w_8)$, $w_4 \in \operatorname{cone}(w_2, w_8)$ and $w_3 \in \operatorname{cone}(w_4, w_7)$, the weights are arranged as follows, where w_7 lies somewhere on the dotted line:



The semiample cone is given as $\operatorname{SAmple}(X) = \operatorname{cone}(w_1, w_3, w_4) \cap \operatorname{cone}(w_2, w_7, w_8)$, i.e. X is of type No. 4.

Case $\mathbf{w_7 \in M_a, w_8 \in M_f}$: We show that this leads to No. 5 in Theorem 3.3.6.

Applying Remark 3.2.5 to γ_{175} and to γ_{138} yields $w_7^2 = 1 = w_8^3$. Thus, the degree matrix $Q = (w_1, \dots, w_8)$ is as follows:

$$Q \ = \ \left(\begin{array}{cc|cc|cc||cc} 1 & d_1 - 1 & 0 & d_1 & 0 & d_1 & w_7^1 & w_8^1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & w_8^2 \\ 0 & 1 & 0 & 1 & 1 & 0 & w_7^3 & 1 \end{array}\right).$$

Note that we have

$$\det(w_7, w_6, w_8) \ = \ \det(w_7, w_3, w_8) + d_1(w_7^3 w_8^2 - 1)\,,$$

i.e. Remark 3.2.5 shows $0 = d_1(w_7^3 w_8^2 - 1)$ $(*)$. Furthermore we have

$$\det(w_7, w_2, w_8) \ = \ \det(w_7, w_3, w_8) + \det(w_7, w_4, w_8) + 1 - w_7^3 w_8^2\,,$$

which together with Remark 3.2.5 shows that $w_7^3 w_8^2 = 2$ holds. Thus, $(*)$ implies $d_1 = 0$. Because of $w_7 \in M_a$, we have

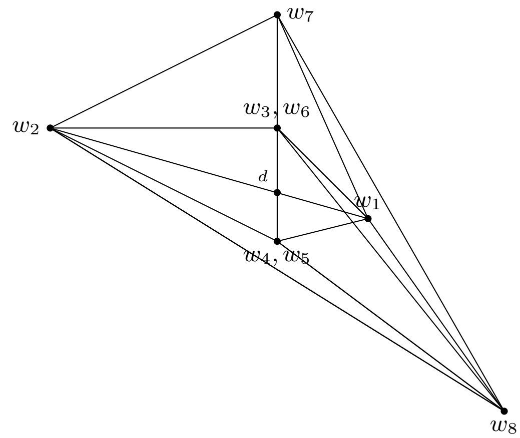
$$0 \ < \ \det(w_1, w_7, w_2) \ = \ 1 - w_7^3\,,$$

i.e. $w_7^3 \leq 0$ holds. Thus we have either $w_7^3 = -1, w_8^2 = -2$ or $w_7^3 = -2, w_8^2 = -1$.

In the first subcase, Remark 3.2.5 applied to γ_{738} and γ_{758} yields $w_7^1 = 0$ and $w_8^1 = 1$. Thus, the degree matrix is given as

$$Q \ = \ \left(\begin{array}{cc|cc|cc||cc} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & -2 \\ 0 & 1 & 0 & 1 & 1 & 0 & -1 & 1 \end{array}\right).$$

Note that we have $w_3 \in \operatorname{cone}(w_4, w_7)$ and that $\operatorname{cone}(w_1, w_4)^\circ \cap \operatorname{cone}(w_3, w_8)^\circ$ is non-empty. Hence the arrangement of weights is as follows:

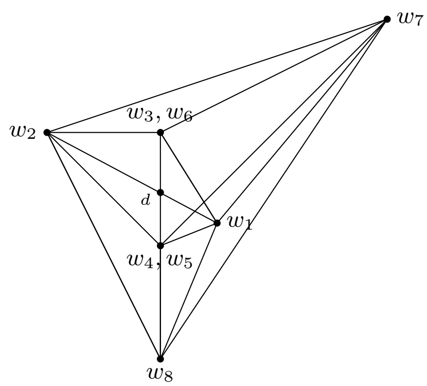


The semiample cone is $\mathrm{SAmple}(X) = \mathrm{cone}(w_1, w_5, w_7) \cap \mathrm{cone}(w_1, w_6, w_8)$, i.e. X is of type No. 5.

We treat the case $w_7^3 = -2, w_8^2 = -1$. Remark 3.2.5 applied to γ_{738} and γ_{758} yields $w_7^1 = 1$ and $w_8^1 = 0$. Thus, the degree matrix is given as

$$Q = \left(\begin{array}{cc|cc|cc||cc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 & 1 & 0 & -2 & 1 \end{array}\right).$$

Note that we have $w_4 \in \mathrm{cone}(w_3, w_8)$, $w_1 \in \mathrm{cone}(w_5, w_7, w_8)$ and $w_3 \in \mathrm{cone}(w_2, w_5, w_7)$. Thus the arrangement of weights is as follows:



The semiample cone is $\mathrm{SAmple}(X) = \mathrm{cone}(w_1, w_5, w_7) \cap \mathrm{cone}(w_1, w_6, w_8)$, i.e. exchanging the second and the third row of Q and renumbering the variables via $(34)(56)(78)$ shows that X is of type No. 5.

Case $\mathbf{w_7 \in M_c, w_8 \in M_c}$: We show that this leads to No. 6 in Theorem 3.3.6.

Applying Remark 3.2.5 to γ_{357} and to γ_{358} yields $w_7^1 = w_8^1 = 1$. Thus, Q is given by

$$Q = \left(\begin{array}{cc|cc|cc||cc} 1 & d_1 - 1 & 0 & d_1 & 0 & d_1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & w_7^2 & w_8^2 \\ 0 & 1 & 0 & 1 & 1 & 0 & w_7^3 & w_8^3 \end{array}\right).$$

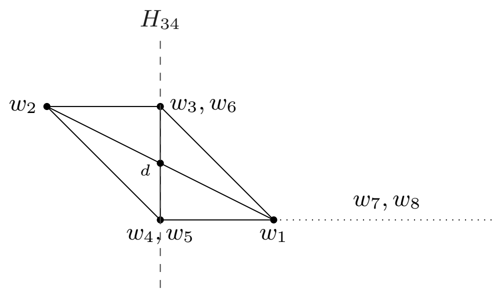
The same remark together with $\gamma_{25i}, \gamma_{24i}, \gamma_{64i}$, $i = 7, 8$, yields

$$0 = (d_1 - 1)w_i^2, \qquad 0 = d_1(w_i^2 - w_i^3), \qquad 0 = d_1(w_i^2 + w_i^3), \qquad i = 7, 8.$$

If $d_1 \neq 0$ held, we would obtain $w_i^2 = w_i^3 = 0$, $i = 7, 8$. Recall that u is contained in $\mathrm{cone}(w_1, w_3, d)$, which implies $u_2 \geq u_3$. But $u \in Q(\gamma_{257})^\circ$ together with $w_7 = (1, 0, 0)$ would yield $u_2 < u_3$, a contradiction. Hence we have $d_1 = 0$ and the first of the above equations shows $w_7^2 = w_8^2 = 0$. Because of $u_2 \geq u_3$ and $u \in Q(\gamma_{257})^\circ$ we have $w_i^3 < 0$, $i = 7, 8$. Thus, $Q = (w_1, \dots, w_8)$ is as follows:

$$Q = \left(\begin{array}{cc|cc|cc||cc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & w_7^3 & w_8^3 \end{array}\right), \qquad w_i^3 < 0, \; i = 7, 8.$$

Possibly after exchanging w_7 and w_8, we may assume that $0 > w_7^3 \geq w_8^3$ holds. We denote by H_{14} and H_{34} the hypersurfaces through w_1, w_4 and through w_3, w_4, respectively. Since $w_7^2 = w_8^2 = 0$ and $w_7^1, w_8^1 > 0$ hold, w_7 and w_8 lie on H_{14} and on the same side of H_{34} as w_1. Thus, the weights are arranged as follows, where w_7 and w_8 lie somewhere on the dotted line:



Since $0 > w_7^3 \geq w_8^3$ and $\mathrm{cone}(w_1, w_3, w_5) \cap \mathrm{cone}(w_2, w_5, w_7) \subseteq \mathrm{cone}(w_3, w_5, w_7)$ hold, the semiample cone of X is the intersection of $\mathrm{cone}(w_1, w_3, w_5)$ and $\mathrm{cone}(w_2, w_5, w_7)$, i.e. X is of type No. 6.

Case $\mathbf{w_7 \in M_c, w_8 \in M_d}$: We show that this leads to No. 7 in Theorem 3.3.6.

Applying Remark 3.2.5 to γ_{357} and to γ_{358} yields $w_7^1 = 1 = w_8^1$. Thus, Q is given by:

$$Q = \left(\begin{array}{cc|cc|cc||cc} 1 & d_1 - 1 & 0 & d_1 & 0 & d_1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & w_7^2 & w_8^2 \\ 0 & 1 & 0 & 1 & 1 & 0 & w_7^3 & w_8^3 \end{array}\right).$$

The same remark together with $\gamma_{257}, \gamma_{247}, \gamma_{647}$ yields

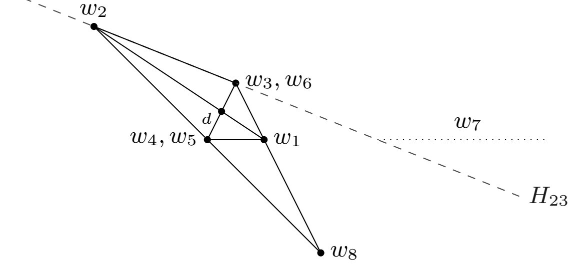
$$0 = (d_1 - 1) w_7^2, \qquad 0 = d_1(w_7^2 - w_7^3), \qquad 0 = d_1(w_7^2 + w_7^3).$$

As in the previous case, we obtain $d_1 = 0$, $w_7^2 = 0$ and $w_7^3 < 0$. Applying Remark 3.2.5 to γ_{283} and γ_{872} yields $w_8^3 = 0$ and $1 + w_8^2 = w_7^3(-w_8^2 - 1)$. We distinguish the subcases $w_8^2 = -1$ and $w_8^2 \neq -1$.

If $w_8^2 = -1$ holds, then $Q = (w_1, \dots, w_8)$ is as follows:

$$Q = \left(\begin{array}{cc|cc|cc||cc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 1 & 0 & w_7^3 & 0 \end{array}\right), \qquad w_7^3 < 0.$$

Note that w_7 lies on the hypersurface through w_1 and w_5. Furthermore, w_7 lies on the opposite side of the hypersurface H_{23} through w_2 and w_3 as w_1. Since in addition $w_1 \in \mathrm{cone}(w_3, w_8)$ and $w_4 \in \mathrm{cone}(w_2, w_8)$ hold, the weights are arranged as follows, where w_7 lies somewhere on the dotted line:



Note that we have $\mathrm{SAmple}(X) = \mathrm{cone}(w_1, w_3, w_5)$ and thus X is of type No. 7.

If $w_8^2 \neq -1$ holds, then $Q = (w_1, \ldots, w_8)$ is as follows:

$$Q \;=\; \left(\begin{array}{cc|cc|cc||cc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & w_8^2 \\ 0 & 1 & 0 & 1 & 1 & 0 & -1 & 0 \end{array}\right).$$

If we swap the second and the third row of the degree matrix Q and renumber the variables via $(3,4)(5,6)(7,8)$, then we can transform this subcase into the previous subcase.

Case $\mathbf{w_7 \in M_c, w_8 \in M_f}$: We show that this leads to No. 8 in Theorem 3.3.6.

Applying Remark 3.2.5 to γ_{357} and to γ_{138} yields $w_7^1 = 1 = w_8^3$. Thus, the degree matrix $Q = (w_1, \ldots, w_8)$ is as follows:

$$Q \;=\; \left(\begin{array}{cc|cc|cc||cc} 1 & d_1 - 1 & 0 & d_1 & 0 & d_1 & 1 & w_8^1 \\ 0 & 1 & 1 & 0 & 0 & 1 & w_7^2 & w_8^2 \\ 0 & 1 & 0 & 1 & 1 & 0 & w_7^3 & 1 \end{array}\right).$$

The same remark together with $\gamma_{257}, \gamma_{247}, \gamma_{647}$ yields

$$0 \;=\; (d_1 - 1)w_7^2\,, \qquad 0 \;=\; d_1(w_7^2 - w_7^3)\,, \qquad 0 \;=\; d_1(w_7^2 + w_7^3)\,.$$

As in the previous case we obtain $d_1 = 0$, $w_7^2 = 0$ and $w_7^3 < 0$.

We first treat the case $l_{7u}(w_8) > 0$. Here we swap the second and the third row of Q and renumber the variables via $(3,4)(5,6)(7,8)$. In this manner, we see that covering collection in the case $w_7 \in M_c, w_8 \in M_f, l_{7u}(w_8) > 0$ coincides with $\mathrm{cov}(u)$ in the case $w_7 \in M_a, w_8 \in M_d$, which we treated above.

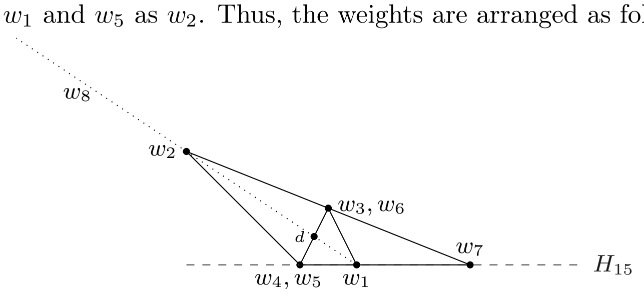
We now treat the case $l_{7u}(w_8) < 0$. Applying Remark 3.2.5 to the faces γ_{178} and γ_{478} yields

$$1 \;=\; -w_7^3 w_8^2\,, \qquad 1 \;=\; w_8^2\,.$$

Hence, $Q = (w_1, \ldots, w_8)$ is as follows:

$$Q \;=\; \left(\begin{array}{cc|cc|cc||cc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & w_8^1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & -1 & 1 \end{array}\right).$$

Note that we have $w_1 \in \mathrm{cone}(w_5, w_7)$ and $w_3 \in \mathrm{cone}(w_2, w_7)$. Furthermore, w_8 lies on the hypersurface H_{12} through w_1 and w_2 and on the same side of the hypersurface H_{15} through w_1 and w_5 as w_2. Thus, the weights are arranged as follows:



Since $\mathrm{cone}(w_1, w_3, w_5) \cap \mathrm{cone}(w_1, w_7, w_8)$ is contained in $\mathrm{cone}(w_1, w_3, w_8)$ we have $\mathrm{SAmple}(X) = \mathrm{cone}(w_1, w_3, w_5) \cap \mathrm{cone}(w_1, w_7, w_8)$. Thus, X is of type No. 8.

Case $\mathbf{w_7 \in M_d, w_8 \in M_d}$: We show that this leads to No. 10 in Theorem 3.3.6.

Applying Remark 3.2.5 to $\gamma_{35i}, i = 7, 8$, yields $w_7^1 = w_8^1 = 1$. Thus, $Q = (w_1, \ldots, w_8)$ is as follows:

$$Q \;=\; \left(\begin{array}{cc|cc|cc||cc} 1 & d_1 - 1 & 0 & d_1 & 0 & d_1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & w_7^2 & w_8^2 \\ 0 & 1 & 0 & 1 & 1 & 0 & w_7^3 & w_8^3 \end{array}\right).$$

The same remark together with $\gamma_{2i3}, \gamma_{62i}$ and γ_{64i} yields

$$0 \;=\; (d_1-1)w_i^3\,, \qquad 0 \;=\; d_1(w_i^2-w_i^3)\,, \qquad 0 \;=\; d_1(w_i^2+w_i^3)\,,$$

for $i=7,8$. We distinguish the subcases $d_1=0$ and $d_1\neq 0$.

In the first subcase, the above relations show that $w_7^3=w_8^3=0$ holds. Possibly after renumbering w_7 and w_8, we may assume that $w_7^2\geq w_8^2$ holds. Exchanging the second and the third row of Q and renumbering the variables via $(3,4)(5,6)$ gives

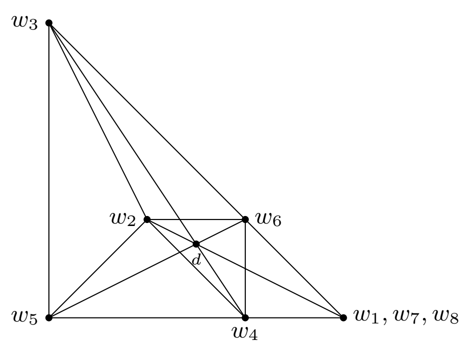
$$Q \;=\; \left(\begin{array}{cc|cc|cc||cc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & w_7^3 & w_8^3 \end{array}\right), \qquad w_7^3 \geq w_8^3\,,$$

$$\mathrm{cov}(u) \;=\; \{\gamma_{135},\gamma_{164},\gamma_{1234},\gamma_{1256},\gamma_{467},\gamma_{274},\gamma_{527},\gamma_{537},\gamma_{468},\gamma_{284},\gamma_{528},\gamma_{538}\}\,.$$

We see that this coincides with the covering collection in the case $w_7, w_8 \in M_c$ which we treated above.

If $d_1\neq 0$ holds, the above relations show that $w_i^2=w_i^3=0$, $i=7,8$, holds. Hence, $Q=(w_1,\ldots,w_8)$ and the arrangement of weights is as follows:

$$Q \;=\; \left(\begin{array}{cc|cc|cc||cc} 1 & d_1-1 & 0 & d_1 & 0 & d_1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right), \quad d_1 > 0\,.$$



Note that we have $\mathrm{SAmple}(X)=\mathrm{cone}(w_1,w_4,w_6)\cap\mathrm{cone}(w_2,w_6,w_7)$, which shows that X is of type No. 10.

Case $\mathbf{w_7 \in M_d, w_8 \in M_f, l_{7u}(w_8) > 0}$: We show that this leads to No. 11 in Theorem 3.3.6.

Applying Remark 3.2.5 to the faces γ_{357} and γ_{138} yields $w_7^1=w_8^3=1$. Thus, the degree matrix $Q=(w_1,\ldots,w_8)$ is as follows:

$$Q \;=\; \left(\begin{array}{cc|cc|cc||cc} 1 & d_1-1 & 0 & d_1 & 0 & d_1 & 1 & w_8^1 \\ 0 & 1 & 1 & 0 & 0 & 1 & w_7^2 & w_8^2 \\ 0 & 1 & 0 & 1 & 1 & 0 & w_7^3 & 1 \end{array}\right).$$

The same remark together with $\gamma_{273}, \gamma_{627}$ and γ_{647} yields

$$0 \;=\; (d_1-1)w_7^3\,, \qquad 0 \;=\; d_1(w_7^2-w_7^3)\,, \qquad 0 \;=\; d_1(w_7^2+w_7^3)\,.$$

We distinguish the cases $d_1=0$ and $d_1\neq 0$.

In the first case, we have $d_1=0$ and $w_7^3=0$. Exchanging the second and the third row of Q and renumbering the variables via $(3,4)(5,6)(7,8)$ gives

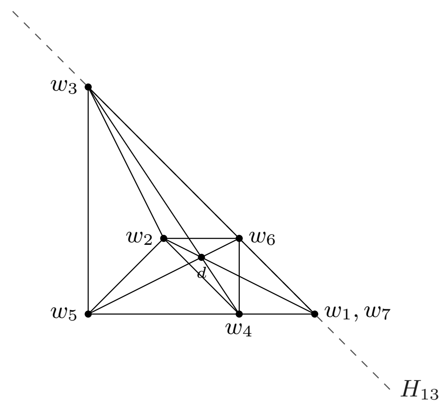
$$Q \;=\; \left(\begin{array}{cc|cc|cc||cc} 1 & -1 & 0 & 0 & 0 & 0 & a & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & b & c \end{array}\right),$$

$$\mathrm{cov}(u) \;=\; \{\gamma_{135},\gamma_{164},\gamma_{1234},\gamma_{1256},\gamma_{468},\gamma_{284},\gamma_{528},\gamma_{538},\gamma_{147},\gamma_{157},\gamma_{784},\gamma_{785}\}\,.$$

We see that this coincides with the covering collection in the case $w_7 \in (a)$, $w_8 \in M_c$, which we treated above.

If $d_1 \neq 0$ holds, the above relations show that $w_7^2 = w_7^3 = 0$ holds. Thus, Q and the arrangement of weights is as follows:

$$Q \;=\; \left(\begin{array}{cc|cc|cc||cc} 1 & d_1-1 & 0 & d_1 & 0 & d_1 & 1 & w_8^1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & w_8^2 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right).$$



Note that w_8 lies on the same side of the hypersurface H_{13} through w_1 and w_3 as w_2. Since $w_6 \in \operatorname{cone}(w_1, w_3)$, $w_4 \in \operatorname{cone}(w_1, w_5)$ and $w_1 = w_7$ hold, the semiample cone of X is

$$\mathrm{SAmple} \;=\; \operatorname{cone}(w_7, w_4, w_6) \;\cap\; \operatorname{cone}(w_6, w_2, w_7) \;\cap\; \operatorname{cone}(w_1, w_6, w_8)\,,$$

i.e. X is of type No. 11.

Case $\mathbf{w_7} \in \mathbf{M_d}, \mathbf{w_8} \in \mathbf{M_f}, \mathbf{l_{7u}(w_8)} < \mathbf{0}$: We show that there is no smooth variety in this case.

As in the previous case applying Remark 3.2.5 to γ_{357}, γ_{138}, $\gamma_{273}, \gamma_{627}$ and γ_{647} yields $w_7^1 = w_8^3 = 1$ as well as

$$0 \;=\; (d_1-1)w_7^3\,, \qquad 0 \;=\; d_1(w_7^2 - w_7^3)\,, \qquad 0 \;=\; d_1(w_7^2 + w_7^3)\,.$$

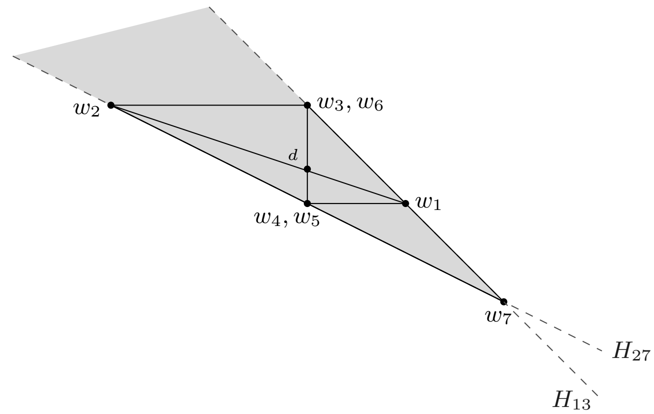
If $w_7^2 = w_7^3 = 0$ holds, the same remark together with γ_{785} and γ_{782} yields $w_8^2 = 1$ and $w_8^2 = 2$, a contradiction. Thus we have $d_1 = w_7^3 = 0$. Note that

$$\det(w_7, w_8, w_2) \;=\; \det(w_7, w_8, w_5) - w_7^2 - 1$$

holds. Remark 3.2.5 applied to γ_{785} and γ_{782} shows that $w_7^2 = -1$ holds. Thus, Remark 3.2.5 applied to γ_{785} yields $w_8^2 = 1 - w_8^1$ and $Q = (w_1, \ldots, w_8)$ is as follows:

$$Q \;=\; \left(\begin{array}{cc|cc|cc||cc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & w_8^1 \\ 0 & 1 & 1 & 0 & 0 & 1 & -1 & 1-w_8^1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right).$$

We have $w_3 + w_7 = w_1$ and $w_2 + w_7 = w_5$. Furthermore, w_8 lies on the same side of the hypersurface H_{13} through w_1 and w_3 as w_5 and on the same side of the hypersurface H_{27} through w_2 and w_7 as w_3. Hence the arrangement of weights is as follows, where w_8 lies somewhere in the gray shaded region:



Recall that we have $\mathrm{SAmple}(X) \subseteq \mathrm{cone}(w_1, w_3, w_8) \cap \mathrm{cone}(w_4, w_7, w_8)$. Since the cone on the right-hand side equals $\mathrm{cone}(w_8)$, this contradicts $\mathbb{Q}$-factoriality of X.

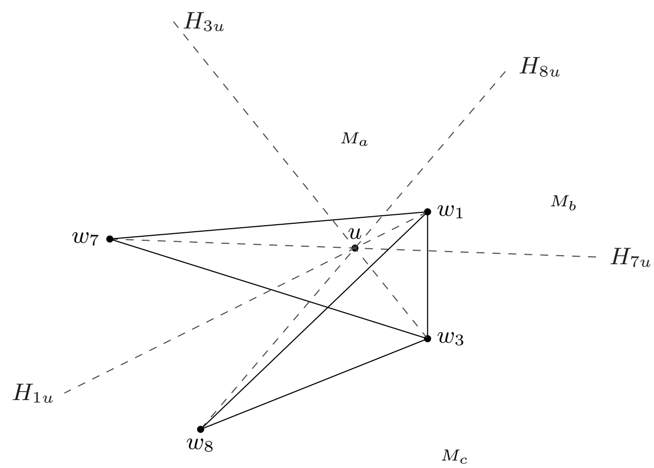
Part 2: To complete the proof, it remains to consider the case $u \notin \tau^\circ$ with $\tau = \mathrm{cone}(w_1, w_2, w_3, w_4, w_5, w_6)$. We first show that $u \notin \tau$ holds. Note that since X is $\mathbb{Q}$-factorial, Remark 3.2.4 shows that $u \notin \mathrm{cone}(w_i, w_j), \mathrm{cone}(w_i)$ holds for all $1 \leq i < j \leq 6$ such that T_i and T_j belong to different monomials of g. Thus after renumbering of variables, $u \in \mathrm{cone}(w_1, w_2)^\circ$ or $u \notin \tau$ holds. We show that $u \in \mathrm{cone}(w_1, w_2)^\circ$ is not possible. Indeed, assume that $u \in \mathrm{cone}(w_1, w_2)^\circ$ holds. In this case Lemma 3.5.4 yields $u \in Q(\gamma_{135})^\circ$. Since X is $\mathbb{Q}$-factorial, $Q(\gamma_{135})$ is three-dimensional. Thus we obtain $u \in \tau^\circ$, a contradiction. We conclude that u is not contained in τ. The definition of $\mathrm{Mov}(X)$ shows that $\mathrm{Mov}(X) \subseteq \tau + \mathrm{cone}(w_7)$ holds. In particular, τ is at least two-dimensional. Note furthermore that – possibly after renumbering of variables – $u \in Q(\gamma_{137})$ holds. Remark 3.2.4 yields $u \in Q(\gamma_{137})^\circ$. We distinguish the two cases $u \in Q(\gamma_{138})$ and $u \notin Q(\gamma_{138})$.

Case $\mathbf{u} \notin \mathbf{Q}(\gamma_{\mathbf{138}})$: We show that this leads to Nos. 16, 17 and 18 in Theorem 3.3.6.

Let $0 \neq l_{iu} \in \mathrm{Hom}(K, \mathbb{Q}),\ i = 1, 3, 7, 8$, be linear forms such that

$$l_{iu}(w_i) = 0 = l_{iu}(u), \qquad l_{iu}(w_3) < 0,\ i = 7, 8, \qquad l_{iu}(w_7) < 0,\ i = 1, 3,$$

holds. After suitable renumbering of variables, the hypersurfaces H_{iu} cut out by l_{iu} are arranged as in the following picture and $\det(w_1, w_3, w_7)$ is strictly negative:



In the figures, M_a, M_b and M_c indicate the following sets of points:

$$\begin{aligned} M_a &= \{x \in K_{\mathbb{Q}};\ l_{3u}(x) > 0,\ l_{8u}(x) > 0\}, \\ M_b &= \{x \in K_{\mathbb{Q}};\ l_{7u}(x) > 0,\ l_{8u}(x) < 0\}, \end{aligned}$$

$$M_c \;=\; \{x \in K_{\mathbb{Q}};\; l_{1u}(x) > 0,\, l_{7u}(x) < 0\}\,.$$

Remark 3.2.4 shows that $l_{\ell u}(w_i)$ is nonzero for $\ell = 7, 8$, $i = 2, 4, 5, 6$. Together with $u \notin \tau$ this shows that w_2, w_4, w_5 and w_6 are contained in $M_a \cup M_b \cup M_c$. Note that if $w_i \in M_a$ holds for some $i \in \{2, 5, 6\}$, then renumbering the variables via $(1i)$ gives $u \in Q(\gamma_{137}) \cap Q(\gamma_{138})$. Since we will treat this case below, we may assume that $w_2, w_5, w_6 \notin M_a$ holds. Hence we have $l_{8u}(w_i) \geq 0$ for all $i = 1, 2, 3, 5, 6, 8$. Since $u \in \mathrm{Mov}(X)^\circ$ holds, we conclude that w_4 is contained in M_a. Similarily, if $w_i \in M_c$ holds for some $i \in \{2, 5, 6\}$, then $u \notin \tau$ gives $u \in Q(\gamma_{i47}) \cap Q(\gamma_{i48})$, i.e. renumbering the variables via $(1i)(34)$ yields $u \in Q(\gamma_{137}) \cap Q(\gamma_{138})$. Since we will treat this case below, we may assume that $w_2, w_5, w_6 \notin M_c$ holds. This gives $w_2, w_5, w_6 \in M_b$. Hence the covering collection is given by

$$\mathrm{cov}(u) \;=\; \{\gamma_{i73},\, \gamma_{i48},\, \gamma_{i78};\;\, i = 1, 2, 5, 6\}\,.$$

Applying Remark 3.2.5 to γ_{173}, γ_{273}, γ_{573} and to γ_{178} yields

$$Q \;=\; \left(\begin{array}{cc|cc|cc||cc} 1 & 1 & 0 & 2 & 1 & 1 & 0 & w_8^1 \\ 0 & d_2 & 0 & d_2 & w_5^2 & d_2 - w_5^2 & 1 & w_8^2 \\ 0 & d_3 & 1 & d_3 - 1 & w_5^3 & d_3 - w_5^3 & 0 & 1 \end{array}\right).$$

The same remark together with the relevant faces γ_{278} and γ_{578} gives $1 = -d_3 w_8^1 + 1$ and $1 = -w_5^3 w_8^1 + 1$. We distinguish the two cases $w_8^1 = 0$ and $w_8^1 \neq 0$, $w_5^3 = d_3 = 0$.

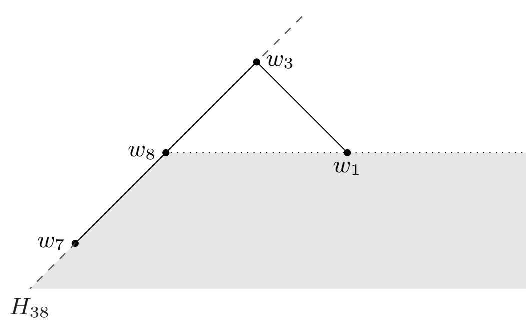
In the first subcase, Remark 3.2.5 applied to γ_{148} and to γ_{248} shows that

$$1 \;=\; -d_3 w_8^2 + d_2 + w_8^2 \qquad \text{and} \qquad 1 \;=\; d_3 w_8^2 - d_2 + w_8^2$$

holds. We conclude $w_8^2 = 1$ and $d_2 = d_3$. Applying again Remark 3.2.5, this time to γ_{548} yields $w_5^2 = w_5^3$. Multiplying Q with an unimodular matrix from the left yields

$$Q \;=\; \left(\begin{array}{cc|cc|cc||cc} 1 & 1 & 0 & 2 & 1 & 1 & 0 & 0 \\ 0 & d_2 & 0 & d_2 & w_5^2 & d_2 - w_5^2 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \end{array}\right).$$

Note that $w_3 + w_7 = w_8$ holds and that w_1, w_2, w_5 and w_6 lie on the same side of the hypersurface H_{38} through w_3 and w_8. Moreover, w_2, w_5 and w_6 lie on the hypersurface H_{18} through w_1 and w_8. Thus the weights are arranged as follows, where w_2, w_5 and w_6 lie somewhere on the dotted line and w_4 somewhere in the gray-shaded area:



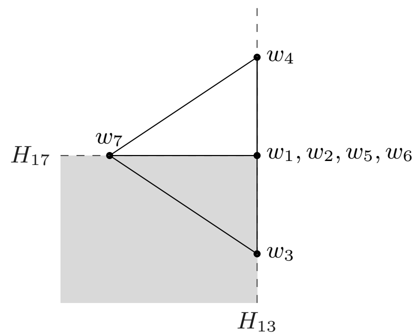
Since $Q(\gamma_{i78}) \subseteq Q(\gamma_{i37})$ holds for $i = 1, 2, 5, 6$, we conclude that the semiample cone of X is the intersection of $Q(\gamma_{i78})$ and $Q(\gamma_{i48})$, $i = 1, 2, 5, 6$. Thus, X is of type No. 16.

In the second subcase, we have $w_8^1 \neq 0$, $w_5^3 = d_3 = 0$. Remark 3.2.5 applied to γ_{148}, to γ_{248} and to γ_{548} shows that we have $d_2 = 1 - w_8^2$ as well as

$$w_8^2(w_8^1 + 2) - w_8^1 - 1 \;=\; 1 \qquad \text{and} \qquad w_5^2(w_8^1 + 2) \;=\; 0\,.$$

If $w_8^1 \neq -2$ holds, then the above equations show that $w_8^2 = 1$, $w_5^2 = 0$ holds. Thus, Q and the arrangement of weights is as follows, where w_8 lies somewhere in the gray-shaded area and where we denote by H_{17} the hypersurface through w_1 and through w_7:

$$Q \;=\; \left(\begin{array}{cc|cc|cc||cc} 1 & 1 & 0 & 2 & 1 & 1 & 0 & w_8^1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right)$$

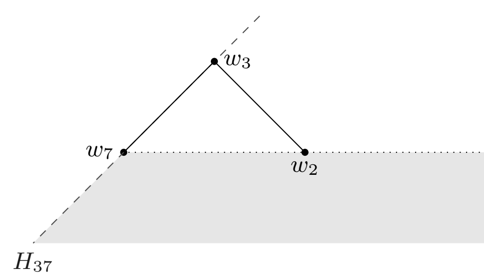


We conclude that the semiample cone of X is the intersection of $Q(\gamma_{178})$, $Q(\gamma_{148})$ and $Q(\gamma_{137})$. Thus, X is of type No. 17.

If $w_8^1 = -2$ holds, then multiplying Q with an unimodular matrix from the left yields

$$Q \;=\; \left(\begin{array}{cc|cc|cc||cc} 1 & 1 & 0 & 2 & 1 & 1 & 0 & -2 \\ w_8^2 & 1 & 0 & 1+w_8^2 & w_5^2+w_8^2 & 1-w_5^2 & 1 & -w_8^2 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right).$$

Note that $w_4 + w_8 = w_7$ holds and that w_1, w_2, w_5 and w_6 lie on the same side of the hypersurface H_{37} through w_3 and w_7. Moreover, w_1, w_5 and w_6 lie on the hypersurface H_{27} through w_2 and w_7. Thus the weights are arranged as follows, where w_2, w_5 and w_6 lie somewhere on the dotted line and w_4 somewhere in the gray-shaded area:



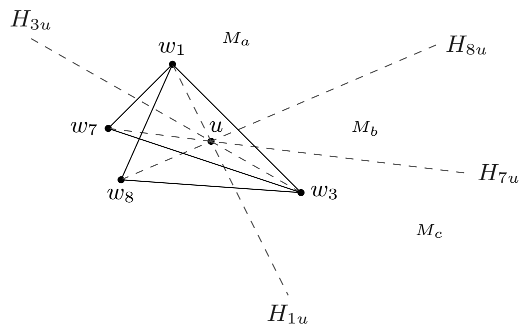
Since $Q(\gamma_{i78}) \subseteq Q(\gamma_{i48})$ holds for $i = 1, 2, 5, 6$, we conclude that the semiample cone of X is the intersection of $Q(\gamma_{i37})$ and $Q(\gamma_{i78})$, $i = 1, 2, 5, 6$. Thus, X is of type No. 18.

Case $\mathbf{u} \in \mathbf{Q}(\gamma_{\mathbf{138}})$: We show that this leads to Nos. 12 – 15 in Theorem 3.3.6.

Remark 3.2.4 shows that $u \in Q(\gamma_{138})^\circ$ holds. Let $0 \neq l_{iu} \in \mathrm{Hom}(K, \mathbb{Q})$, $i = 1, 3, 7, 8$, be linear forms such that

$$l_{iu}(w_i) \;=\; 0 \;=\; l_{iu}(u)\,, \qquad l_{iu}(w_3) \;<\; 0\,,\; i \;=\; 7, 8\,, \qquad l_{iu}(w_7) \;<\; 0\,,\; i = 1, 3\,,$$

holds. After suitable renumbering of variables, the weights w_1, w_3, w_7 and w_8 are arranged as in the following picture and $\det(w_1, w_3, w_7)$ is strictly positive:



In the figures, M_a, M_b and M_c indicate the following sets of points:

$$\begin{aligned} M_a &= \{x \in K_{\mathbb{Q}};\ l_{3u}(x) > 0,\ l_{8u}(x) > 0\}, \\ M_b &= \{x \in K_{\mathbb{Q}};\ l_{7u}(x) > 0,\ l_{8u}(x) < 0\}, \\ M_c &= \{x \in K_{\mathbb{Q}};\ l_{1u}(x) > 0,\ l_{7u}(x) < 0\}. \end{aligned}$$

Note that the faces $\gamma_{i\ell}$ are $\mathfrak{F}$-faces for all $i = 7, 8$, $\ell = 2, 4, 5, 6$. Thus Remark 3.2.4 together with $u \notin \tau$ shows that w_ℓ is contained in $M_a \cup M_b \cup M_c$ for $\ell = 2, 4, 5, 6$. After renumbering of weights, we may assume that $w_5 \in M_a$ or $w_5 \in M_b$ holds. Note that $u \in \mathrm{Mov}(X)^\circ$ shows that one of the weights w_i, $i \in \{2, 4, 5, 6\}$, lies in M_b or one weight lies in M_a and a second one in M_c. Furthermore, the homogeneity of g restricts the possible arrangements of w_2, w_4, w_5, w_6 in $M_a \cup M_b \cup M_c$. For instance if $w_2 \in M_a$ holds, then we obtain $\deg(g) \in M_a$ and thus $w_4, w_5 \in M_a$ or $w_4, w_6 \in M_a$ holds. After suitable renumbering of variables, the weights w_2, w_4, w_5, w_6 are arranged as in the following table. To see this, note that the first part of the table contains all constellations with $w_5 \in M_a$; in the second part it remains to consider the constellations $w_5, w_6 \in M_b$. As a matter of convenience, we list the indices i, j, k of the faces γ_{ijk} in an order such that (w_i, w_j, w_k) is positively orientated, i.e. $\det(w_i, w_j, w_k) > 0$ holds.

case	w_2	w_4	w_5	w_6	$\mathrm{cov}(u) \setminus \{\gamma_{137}, \gamma_{138}\}$ contains
(i)	M_a	M_a	M_a	M_b	$\gamma_{238}, \gamma_{538}, \gamma_{168}, \gamma_{268}, \gamma_{468}$
(ii)	M_a	M_a	M_a	M_c	$\gamma_{238}, \gamma_{538}, \gamma_{168}, \gamma_{268}, \gamma_{468}$
(iii)	M_b	M_a	M_a	M_a	$\gamma_{538}, \gamma_{638}, \gamma_{428}, \gamma_{528}, \gamma_{628}$
(iv)	M_b	M_a	M_a	M_b	$\gamma_{538}, \gamma_{237}, \gamma_{537}, \gamma_{637}, \gamma_{287}, \gamma_{687}, \gamma_{168}, \gamma_{528}, \gamma_{428}, \gamma_{468}$
(v)	M_b	M_a	M_a	M_c	$\gamma_{538}, \gamma_{237}, \gamma_{537}, \gamma_{168}, \gamma_{428}, \gamma_{267}, \gamma_{468}$
(vi)	M_b	M_b	M_a	M_b	$\gamma_{148}, \gamma_{168}, \gamma_{528}, \gamma_{538}, \gamma_{548}$
(vii)	M_b	M_b	M_a	M_c	$\gamma_{148}, \gamma_{168}, \gamma_{528}, \gamma_{538}, \gamma_{548}$
$(viii)$	M_c	M_a	M_a	M_a	$\gamma_{538}, \gamma_{638}, \gamma_{428}, \gamma_{528}, \gamma_{628}$
(ix)	M_c	M_a	M_a	M_b	$\gamma_{537}, \gamma_{637}, \gamma_{427}, \gamma_{527}, \gamma_{627}$
(x)	M_c	M_a	M_a	M_c	$\gamma_{538}, \gamma_{537}, \gamma_{168}, \gamma_{167}, \gamma_{527}, \gamma_{528}, \gamma_{427}, \gamma_{428}, \gamma_{467}, \gamma_{468}$
(xi)	M_c	M_b	M_a	M_b	$\gamma_{537}, \gamma_{637}, \gamma_{427}, \gamma_{527}, \gamma_{627}$
	M_c	M_b	M_a	M_c	renumbering of variables via (13)(24)(56)(78) yields case (v)
	M_c	M_c	M_a	M_c	renumbering of variables via (13)(56)(78) yields case (ii)
	M_b	M_a	M_b	M_b	renumbering of variables via (13)(24)(78) yields case (xiv)
(xii)	M_b	M_b	M_b	M_b	$\gamma_{237}, \gamma_{537}, \gamma_{637}, \gamma_{148}, \gamma_{158}, \gamma_{168}, \gamma_{287}, \gamma_{487}, \gamma_{587}, \gamma_{687}$
$(xiii)$	M_c	M_a	M_b	M_b	$\gamma_{158}, \gamma_{168}, \gamma_{428}, \gamma_{458}, \gamma_{468}$

(xiv)	M_c	M_b	M_b	M_b	$\gamma_{537}, \gamma_{637}, \gamma_{427}, \gamma_{527}, \gamma_{627}$

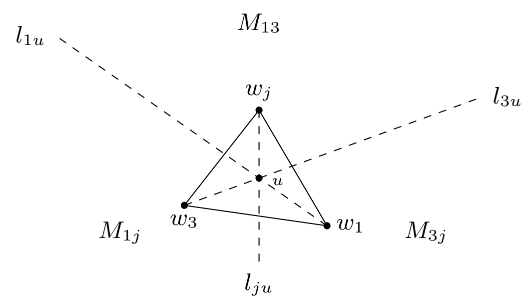
To complete the proof, we apply Remark 3.2.4 and Remark 3.2.5 to these cases and show that we end up with one of the varieties of type Nos. 12–15 in the table of Theorem 3.3.6. Note that Lemma 3.1.6 shows that the resulting varieties are smooth.

Cases (iii), (viii), (ix), (xi), (xiv): We show that there is no smooth variety in these cases.

In all these cases, there is $j \in \{7, 8\}$ such that $\gamma_{13j}, \gamma_{53j}, \gamma_{63j}, \gamma_{42j}, \gamma_{52j}$ and γ_{62j} are relevant faces. Remark 3.2.5 together with γ_{13j} shows that we may assume $(w_1, w_3, w_j) = (e_1, e_2, e_3)$ with the canonical base vectors e_1, e_2, e_3 of $\mathbb{Q}^3$. By $l_{iu} \in \mathrm{Hom}(K, \mathbb{Q}),\ i = 1, 3, j$, we denote linear forms with

$$l_{iu}(w_i) \ = \ l_{iu}(u) \ = \ 0, \ i \ = \ 1, 3, j \qquad \text{and} \qquad l_{1u}(w_j), l_{3u}(w_1), l_{ju}(w_3) \ > \ 0\,.$$

The weights w_i and linear forms $l_{iu},\ i = 1, 3, j$ are arranged as follows:



In the picture, M_{13}, M_{3j}, M_{1j} indicate the set of points between the respective hyperplanes cut out by $l_{iu} = 0,\ i = 1, 3, j$. For instance, we have

$$M_{13} \ = \ \{x \in K_{\mathbb{Q}};\ l_{1u}(x) > 0,\, l_{3u}(x) < 0\}\,.$$

Since $u \in Q(\gamma_{53j})^\circ \cap Q(\gamma_{63j})^\circ$ holds, the weights w_5 and w_6 are contained in M_{3j}. This means that $\det(w_i, w_3, w_j),\ i = 5, 6$, is strictly positive. Thus Remark 3.2.5 yields $w_5^1 = 1 = w_6^1$ and we obtain

$$(w_1, w_2 | w_3, w_4 | w_5, w_6 || w_j) \ = \ \left(\begin{array}{cc|cc|cc||c} 1 & 1 & 0 & 2 & 1 & 1 & 0 \\ 0 & d_2 & 1 & d_2 & w_5^2 & d_2 - w_5^2 & 0 \\ 0 & d_3 & 0 & d_3 - 1 & w_5^2 & d_3 - w_5^3 & 1 \end{array}\right),$$

where $d = (2, d_2, d_3)$ denotes the degree of g. Since $u \in Q(\gamma_{52j})^\circ$ holds, we have $l_{ju}(w_2) > 0$. Thus, $\det(w_5, w_2, w_j)$ and $\det(w_6, w_2, w_j)$ are strictly positive. Hence Remark 3.2.5 applied to γ_{52j} and to γ_{62j} yields $1 = d_2 - w_5^2$ and $1 = w_5^2$, i.e. we obtain $d_2 = 2$. Note that $w_5, w_6 \in M_{3j}$ shows that also d and hence w_4 are contained in M_{3j}. We conclude $\det(w_4, w_2, w_j) > 0$. Remark 3.2.5 yields $1 = \det(w_4, w_2, w_j) = 2$, a contradiction. Hence there are no smooth varieties in Cases (iii), (viii), (ix), (xi) and (xiv).

Case (xiii): We show that there is no smooth variety in this case.

We renumber the variables via (13)(24). Then $\gamma_{138}, \gamma_{358}, \gamma_{368}, \gamma_{248}, \gamma_{258}$ and γ_{268} are relevant faces. This shows that the proof in this case is analogous to the previous proof of Cases (iii), (viii), (ix), (xi) and (xiv).

For the remaining part of the proof, we apply Remark 3.2.5 to γ_{137} and to γ_{138} and obtain

$$Q = \left(\begin{array}{cc|cc|cc||cc} 1 & d_1-1 & 0 & d_1 & w_5^1 & d_1-w_5^1 & w_7^1 & 0 \\ 0 & d_2 & 1 & d_2-1 & w_5^2 & d_2-w_5^2 & w_7^2 & 0 \\ 0 & d_3 & 0 & d_3 & w_5^3 & d_3-w_5^3 & 1 & 1 \end{array}\right),$$

where $d = (d_1, d_2, d_3)$ denotes the degree of g.

Cases (i) and (ii): We show that there is no smooth variety in this case.

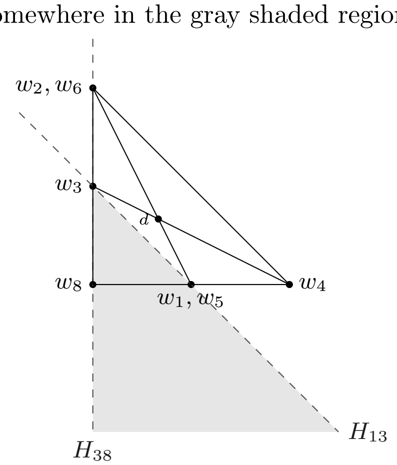
Remark 3.2.5 together with $\gamma_{238}, \gamma_{538}$ and γ_{268} yields $d_1 = 2$, $w_5^1 = 1$ and $w_5^2 = -1$. Thus, the same remark applied to γ_{168} and to γ_{468} shows that $d_2 + 1 = 1 = d_2 + 3$ holds, a contradiction. Hence there are no smooth varieties in these cases.

Case (iv): We show that this case leads to No. 12 in Theorem 3.3.6.

Remark 3.2.5 together with $\gamma_{538}, \gamma_{168}$ and γ_{428} yields $w_5^1 = 1, w_5^2 = d_2 - 1$ and $d_2 = 2 - d_1$. Thus, the same remark applied to γ_{528} and γ_{468} shows that $d_1(d_1 - 3) = -2$ as well as $d_1(d_1 - 1) = 0$ hold. We obtain $d_1 = 1$, $d_2 = 1$ and $w_5^2 = 0$. Remark 3.2.5 applied to γ_{687}, γ_{237}, γ_{637} and γ_{537} yields $w_7^1 = 1$, $d_3 = -1$ and $w_5^3 = 0$. Thus, Q is as follows:

$$Q = \left(\begin{array}{cc|cc|cc||cc} 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & w_7^2 & 0 \\ 0 & -1 & 0 & -1 & 0 & -1 & 1 & 1 \end{array}\right),$$

We have $w_6 + w_8 = w_3$ and $w_4 + w_8 = w_1$. Furthermore, w_7 lies on the same side of the hypersurface H_{13} through w_1 and w_3 as w_8 and on the same side of the hypersurface H_{38} through w_3 and w_8 as w_1. Thus the arrangement of weights is as follows, where w_7 lies somewhere in the gray shaded region:



The covering collection of X consists of the cones listed in the above table. Since

$$\operatorname{cone}(w_1, w_3, w_8) \cap \operatorname{cone}(w_2, w_3, w_7) \subseteq \operatorname{cone}(w_1, w_3, w_7)$$

holds, the semiample cone of X is the intersection of $Q(\gamma_{138})$ and $Q(\gamma_{237})$, i.e. X is of type No. 12.

Case (v): We show that there is no smooth variety in this case.

Remark 3.2.5 together with $\gamma_{538}, \gamma_{168}$ and γ_{428} yields $w_5^1 = 1, w_5^2 = d_2 - 1$ and $d_2 = 2 - d_1$. Thus, the same remark applied to γ_{468} shows that $d_1(d_1 - 1) = 0$ holds. We distinguish the cases $d_1 = 0$ and $d_1 = 1$. If $d_1 = 0$ holds, we obtain

$$Q = \left(\begin{array}{cc|cc|cc||cc} 1 & -1 & 0 & 0 & 1 & -1 & w_7^1 & 0 \\ 0 & 2 & 1 & 1 & 1 & 1 & w_7^2 & 0 \\ 0 & d_3 & 0 & d_3 & w_5^3 & d_3-w_5^3 & 1 & 1 \end{array}\right).$$

Remark 3.2.5 together with γ_{237} and γ_{537} shows that

$$d_3 w_7^1 \;=\; -2 \qquad \text{and} \qquad w_5^3 w_7^1 \;=\; 0$$

hold. This gives $w_5^3 = 0$. Hence Remark 3.2.5 together with γ_{267} yields $d_3 w_7^1 = 0$, a contradiction to the first one of the above relations. Now we treat the case $d_1 = 1$. Here, Q is given by

$$Q \;=\; \left(\begin{array}{cc|cc|cc||cc} 1 & 0 & 0 & 1 & 1 & 0 & w_7^1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & w_7^2 & 0 \\ 0 & d_3 & 0 & d_3 & w_5^3 & d_3 - w_5^3 & 1 & 1 \end{array}\right).$$

Remark 3.2.5 together with γ_{237} and γ_{537} shows that

$$d_3 w_7^1 \;=\; -1 \qquad \text{and} \qquad w_5^3 w_7^1 \;=\; 0$$

hold. This gives $w_5^3 = 0$. Hence $w_2 = w_6$ holds. Thus $Q(\gamma_{267})$ is a two-dimensional cone, which contradicts Remark 3.2.4. Hence there are no smooth varieties in this case.

Cases (vi) and (vii): We show that there are no smooth varieties in these cases.

Remark 3.2.5 together with $\gamma_{148}, \gamma_{538}$ and γ_{168} yields $d_2 = 2$, $w_5^1 = 1$ and $w_5^2 = 1$. Thus, the same remark applied to γ_{528} and to γ_{548} shows that $3 - d_1 = 1 - d_1$ holds, a contradiction. Hence there are no smooth varieties in these cases.

Case (x): We show that this case leads to Nos. 13 and 14 in Theorem 3.3.6.

Remark 3.2.5 together with $\gamma_{538}, \gamma_{168}$ and γ_{428} yields $w_5^1 = 1$, $w_5^2 = d_2 - 1$, and $d_2 = 2 - d_1$. Thus, the same remark applied to γ_{528} and γ_{468} shows that $d_1^2 - 3d_1 + 3 = 1 = d_1^2 - d_1 + 1$ holds. We conclude $d_1 = 1 = d_2$ and $w_5^2 = 0$, i.e. Q is given as

$$Q \;=\; \left(\begin{array}{cc|cc|cc||cc} 1 & 0 & 0 & 1 & 1 & 0 & w_7^1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & w_7^2 & 0 \\ 0 & d_3 & 0 & d_3 & w_5^3 & d_3 - w_5^3 & 1 & 1 \end{array}\right).$$

Remark 3.2.5 applied to γ_{537}, to γ_{527}, to γ_{167} and to γ_{427} yields

$$w_5^3 w_7^1 \;=\; 0\,, \qquad w_5^3 w_7^2 \;=\; 0\,, \qquad d_3 w_7^1 \;=\; 0\,, \qquad d_3 w_7^2 \;=\; 0\,.$$

We distinguish the cases $d_3 = w_5^3 = 0$ and $w_7^1 = w_7^2 = 0$. In the first case Q is as follows:

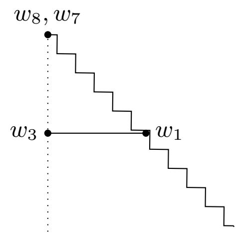
$$Q \;=\; \left(\begin{array}{cc|cc|cc||cc} 1 & 0 & 0 & 1 & 1 & 0 & w_7^1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & w_7^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right).$$

We have $w_1 = w_4 = w_5$ and $w_2 = w_3 = w_6$. Furthermore, w_7 lies on the same side of the hypersurface H_{13} through w_1 and w_3 as w_8. Thus, $\mathrm{SAmple}(X) = Q(\gamma_{137}) \cap Q(\gamma_{138})$ holds and X is of type No. 13.

In the second case, we have and $w_7^1 = w_7^2 = 0$. Hence Q is as follows:

$$Q \;=\; \left(\begin{array}{cc|cc|cc||cc} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & d_3 & 0 & d_3 & w_5^3 & d_3 - w_5^3 & 1 & 1 \end{array}\right).$$

Note that w_4 and w_5 lie on the hypersurface H_{17} through w_1 and w_7. Moreover, w_2 and w_6 lie on the hypersurface H_{37} through w_3 and w_7. Furthermore, the weights w_2, w_3 and w_6 are on the same side of H_{17}. The same holds for H_{37} and the weights w_1, w_4 and w_5. This shows that the arrangement of weights is as follows, where w_2 and w_6 lie somewhere on the dotted line and w_4 and w_5 somewhere on the zigzag line:



We have $\mathrm{SAmple}(X) = Q(\gamma_{137}) \cap Q(\gamma_{167}) \cap Q(\gamma_{537}) \cap Q(\gamma_{427}) \cap Q(\gamma_{527}) \cap Q(\gamma_{467})$, which shows that X is of type No. 14.

Case (xii): We show that this case leads to No. 15 in Theorem 3.3.6.

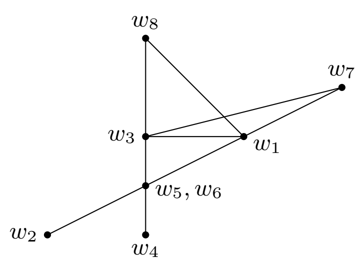
Remark 3.2.5 together with γ_{148} and γ_{158} yields $d_2 = 2$ and $w_5^2 = 1$. Thus, Q is given as

$$Q \;=\; \left(\begin{array}{cc|cc|cc||cc} 1 & d_1-1 & 0 & d_1 & w_5^1 & d_1-w_5^1 & w_7^1 & 0 \\ 0 & 2 & 1 & 1 & 1 & 1 & w_7^2 & 0 \\ 0 & d_3 & 0 & d_3 & w_5^3 & d_3-w_5^3 & 1 & 1 \end{array}\right).$$

Since $\det(w_2, w_8, w_7) = \det(w_4, w_8, w_7) + w_7^1 + w_7^2$ holds, Remark 3.2.5 yields $w_7^1 = -w_7^2$. Thus, Remark 3.2.5 applied again to γ_{287} shows that $w_7^2(-d_1 - 1) = 1$ holds. This gives $w_7^2 = 1, d_1 = -2$ or $w_7^2 = -1, d_1 = 0$. In the first case, Remark 3.2.5 applied to γ_{687} and to γ_{587} yields $0 = w_5^1 = -2$, a contradiction. Thus, $w_7^2 = -1, d_1 = 0$ holds. Remark 3.2.5 applied to $\gamma_{237}, \gamma_{587}$ and γ_{537} yields $d_3 = -2$, $w_5^1 = 0$ and $w_5^3 = -1$. Hence Q is as follows:

$$Q \;=\; \left(\begin{array}{cc|cc|cc||cc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 & 1 & 1 & -1 & 0 \\ 0 & -2 & 0 & -2 & -1 & -1 & 1 & 1 \end{array}\right).$$

Note that we have $w_5 + w_8 = w_3$, $w_1 + w_2 = 2w_5$ and $w_5 + w_7 = w_1$. Thus the arrangement of weights is as follows:



We have $\mathrm{SAmple}(X) = Q(\gamma_{137}) \cap Q(\gamma_{138})$, which shows that X is of type No. 15. $\square$

3.10. Proof of Proposition 3.10.1

In this section we give a description of all smooth four-dimensional intrinsic quadrics of Picard number three whose Cox ring contains three free variables.

Proposition 3.10.1. *Let X be a four-dimensional intrinsic quadric of Picard number three with Cox ring*

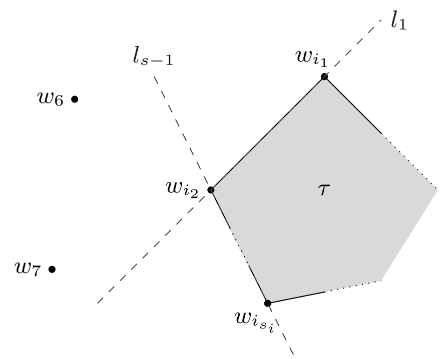
$$\mathcal{R}(X) \;=\; \mathbb{K}[T_1, \ldots, T_8]/\langle g \rangle, \qquad g \;=\; T_1T_2 + T_3T_4 + T_5^2.$$

If X is smooth, then we have $\mathrm{Cl}(X) = \mathbb{Z}^3$ and X is isomorphic to one of the varieties 19 – 37 in the table of Theorem 3.3.6.

Proof. By $w_1, \ldots, w_8$ we denote the degrees of the variables $T_1, \ldots, T_8$ and by u an ample Weil divisor class. Lemma 3.5.16 shows that $u \notin \tau := Q(\gamma_{1234})$ holds. Note that $\tau \not\ni u \in \mathrm{Mov}(X)^\circ$ shows that at least two of the weights w_6, w_7, w_8 are not contained in τ. Possibly after renumbering of variables, we have $w_6, w_7 \notin \tau$. The definition of $\mathrm{Mov}(X)$ implies that $\mathrm{Mov}(X) \subseteq \tau + \mathrm{cone}(w_6, w_7)$ holds. Consider $i_1, \ldots, i_{s_i} \in \{1, \ldots, 4\}$ such that

$$\tau + \mathrm{cone}(w_6, w_7) \setminus \tau \ \subseteq \ \mathrm{cone}(w_{i_1}, w_{i_2}, w_6, w_7) \cup \ldots \cup \mathrm{cone}(w_{i_{s-1}}, w_{i_s}, w_6, w_7) \ (*)$$

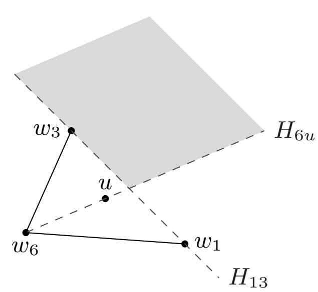
as well as $\mathrm{cone}(w_{i_j}, w_{i_{j+1}}, w_6, w_7) \cap \tau = \mathrm{cone}(w_{i_j}, w_{i_{j+1}})$ and $l_j(w_{i_j}) = l_j(w_{i_{j+1}}) = 0$, $l_j(w_\ell) \leq 0$, $\ell = 1, \ldots, 5$, holds for linear forms $l_j \in \mathrm{Hom}(K, \mathbb{Q})$.



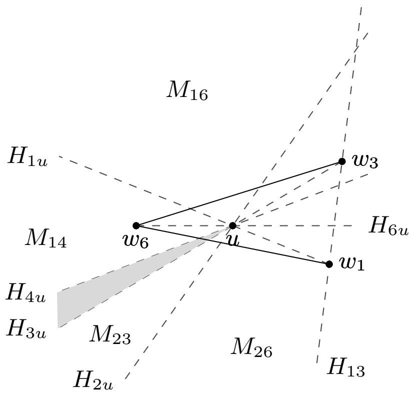
If τ is three-dimensional, together with the homogeneity of g, this further shows that for all $1 \leq j \leq s_i - 1$, T_{i_j} and $T_{i_{j+1}}$ belong to different monomials of g. If τ is two- or one-dimensional, we can choose the i_j in a manner such that for all $1 \leq j \leq s_i - 1$, T_{i_j} and $T_{i_{j+1}}$ belong to different monomials of g. Thus all faces of $\gamma_{i_j, i_{j+1}, 6, 7}$, where j is odd, are $\mathfrak{F}$-faces.

For $\ell = 6, 7, 8$ we set $\kappa_{\ell j} := \gamma_{i_j, i_{j+1}, \ell}$. We now show that we may assume that there is $1 \leq j_0 \leq s_i - 1$ such that $\kappa_{6 j_0}$ is a relevant face. If there is $1 \leq j_0 \leq s_i - 1$ such that $u \in Q(\kappa_{6 j_0})$ or $u \in Q(\kappa_{7 j_0})$ holds, then Remark 3.2.4 shows that $\kappa_{j_0 6}$ or $\kappa_{7 j_0}$ is a relevant face and thus suitable renumbering of variables yields $u \in Q(\kappa_{6 j_0})^\circ$. If $u \notin Q(\kappa_{6j}) \cup Q(\kappa_{7j})$ holds for all $1 \leq j \leq s_i - 1$, then $\tau \not\ni u \in \mathrm{Mov}(X)^\circ$ shows that $w_8 \notin \tau$ and $u \in Q(\kappa_{8 j_0})$ holds for some $1 \leq j_0 \leq s_i - 1$. Again Remark 3.2.4 yields $u \in Q(\kappa_{8 j_0})^\circ$.

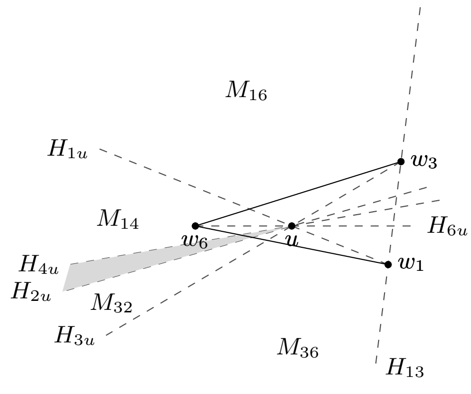
Suitable renumbering of variables yields $j_0 = 1$, $j_0 + 1 = 3$, $\gamma_{136} \in \mathrm{rlv}(u)$ and there is a linear form $l_{13} \in \mathrm{Hom}(K, \mathbb{Q})$ with $l_{13}(w_\ell) \leq 0$, $\ell = 1, \ldots, 5$, as well as $l_{13}(w_6), l_{13}(w_7) \geq 0$. Furthermore we may assume that $\det(w_1, w_3, w_6)$ is strictly positive. Remark 3.2.5 applied to γ_{136} yields $1 = \det(w_1, w_3, w_6)$. Let $0 \neq l_{6u} \in \mathrm{Hom}(K, \mathbb{Q})$ be a linear form such that $l_{6u}(w_6) = 0 = l_{6u}(u)$ and $l_{6u}(w_1) > 0$ holds. Lemma 3.5.9 shows that we may assume that $l_{6u}(w_2)$ and $l_{6u}(w_4)$ are strictly negative. Thus, the weights are arranged as follows, where w_2 and w_4 lie somewhere in the gray-shaded area:



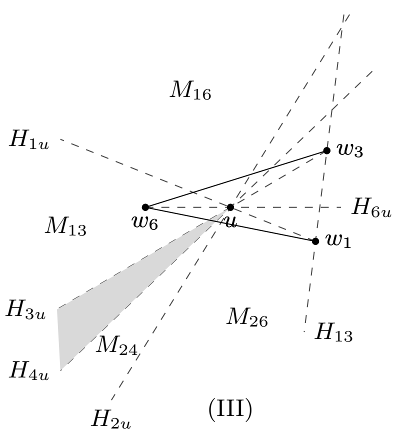
Let $0 \neq l_{iu} \in \mathrm{Hom}(K, \mathbb{Q})$ be linear forms such that $l_{iu}(w_i) = 0 = l_{iu}(u)$ and $l_{iu}(w_6) > 0$ holds for $i = 1, 2, 3, 4$. If $l_{4u}(w_3) > 0$ holds, then homogeneity of g yields $l_{4u}(w_2) > 0$ and if $l_{4u}(w_3) \leq 0$ holds, homogeneity of g shows that $l_{3u}(w_2) > 0$ holds. Thus there remain the following four possibilities for l_{2u} and l_{4u}:



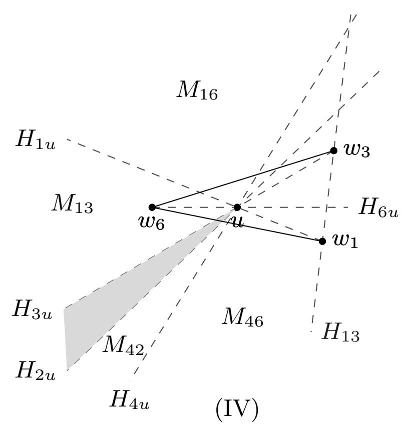
(I)



(II)



(III)



(IV)

In the pictures, M_{ij} indicate the set of points between the respective hyperplanes H_{iu} and H_{ju} cut out by $l_{iu} = 0$ and by $l_{ju} = 0$, i.e. we have

$$
\begin{aligned}
M_{16} &= \{x \in K_{\mathbb{Q}};\ l_{1u}(x) < 0,\ l_{6u}(x) < 0\},\\
M_{1a} &= \{x \in K_{\mathbb{Q}};\ l_{1u}(x) > 0,\ l_{au}(x) > 0\},\ a = 3, 4,\\
M_{a6} &= \{x \in K_{\mathbb{Q}};\ l_{iu}(x) < 0,\ l_{6u}(x) > 0\},\ a = 2, 3, 4,\\
M_{ab} &= \{x \in K_{\mathbb{Q}};\ l_{au}(x) > 0,\ l_{bu}(x) < 0\},\ (a, b) \in \{(2, 3), (2, 4)\}.
\end{aligned}
$$

Furthermore, Lemma 3.5.9 shows that w_7 and w_8 do not lie in the gray-shaded areas. Remark 3.2.4 applied to the faces γ_{ab}, $a = 1, 2, 3, 4, 6$, $b = 7, 8$, shows that w_7 and w_8 lie in one of the above defined sets M_{ij}. Note that $u \in \mathrm{Mov}(X)^\circ$ implies that we may assume $l_{1u}(w_7) > 0$. In particular, $w_7 \notin M_{16}$ holds.

After suitable renumbering of variables, the weights w_7, w_8 are arranged as in the following table. To see this, note that picture (I) and (III) yield the same covering collections, i.e. there is no need to distinguish these cases. Similarily, picture (I) and (II) need to be distinguished only if $w_\ell \in M_{23}$ or $w_\ell \in M_{32}$ holds for some $7 \leq \ell \leq 8$. Picture (III) and (IV) yield the same varieties if $w_7, w_8 \notin M_{24}$ and $w_7, w_8 \notin M_{42}$ holds. Furthermore, note that certain combinations are not possible because of $u \in \mathrm{Mov}(X)^\circ$; for instance $w_7, w_8 \in M_{14}$ would lead to $u \notin \mathrm{Mov}(X)^\circ$. Moreover, Lemma 3.1.5 implies that $u \notin Q(\gamma_{678})^\circ$ holds. As a matter of convenience, we list the indices i, j, k of the faces γ_{ijk} in an order such that (w_i, w_j, w_k) is positively orientated, i.e. $\det(w_i, w_j, w_k)$ is strictly positive.

case	picture	w_7	w_8	$\mathrm{cov}(u) \setminus \{\gamma_{136}, \gamma_{146}, \gamma_{1256}\}$
(i)	(I)	M_{14}	M_{23}	$\gamma_{368}, \gamma_{468}, \gamma_{283}, \gamma_{284}, \gamma_{1258}, \gamma_{137}, \gamma_{147}, \gamma_{1257}, \gamma_{378}, \gamma_{478}$
(ii)	(I)	M_{14}	M_{26}	$\gamma_{368}, \gamma_{468}, \gamma_{268}, \gamma_{137}, \gamma_{147}, \gamma_{1257}, \gamma_{378}, \gamma_{478}, \gamma_{278}$
(iii)	(I)	M_{23}	M_{23}	$\gamma_{367}, \gamma_{467}, \gamma_{327}, \gamma_{427}, \gamma_{1257}, \gamma_{368}, \gamma_{468}, \gamma_{328}, \gamma_{428}, \gamma_{1258}$
(iv)	(I)	M_{23}	M_{26}	$\gamma_{367}, \gamma_{467}, \gamma_{327}, \gamma_{427}, \gamma_{1257}, \gamma_{368}, \gamma_{468}, \gamma_{268}, \gamma_{278}$
(v)	(I)	M_{23}	M_{16}	$\gamma_{367}, \gamma_{467}, \gamma_{327}, \gamma_{427}, \gamma_{1257}, \gamma_{187}, \gamma_{387}, \gamma_{487}, \gamma_{186}$
(vi)	(I)	M_{26}	M_{16}	$\gamma_{267}, \gamma_{367}, \gamma_{467}, \gamma_{187}, \gamma_{287}, \gamma_{387}, \gamma_{478}, \gamma_{186}$
(vii)	(II)	M_{14}	M_{32}	$\gamma_{238}, \gamma_{3458}, \gamma_{138}, \gamma_{268}, \gamma_{468}, \gamma_{137}, \gamma_{1257}, \gamma_{147}, \gamma_{278}, \gamma_{478}$
$(viii)$	(II)	M_{32}	M_{32}	$\gamma_{237}, \gamma_{3457}, \gamma_{137}, \gamma_{267}, \gamma_{467}, \gamma_{238}, \gamma_{3458}, \gamma_{138}, \gamma_{268}, \gamma_{468}$
(ix)	(II)	M_{32}	M_{36}	$\gamma_{237}, \gamma_{3457}, \gamma_{137}, \gamma_{267}, \gamma_{467}, \gamma_{368}, \gamma_{268}, \gamma_{468}, \gamma_{378}$
(x)	(II)	M_{32}	M_{16}	$\gamma_{237}, \gamma_{3457}, \gamma_{137}, \gamma_{267}, \gamma_{467}, \gamma_{186}, \gamma_{187}, \gamma_{287}, \gamma_{487}$
(xi)	(IV)	M_{13}	M_{42}	$\gamma_{148}, \gamma_{248}, \gamma_{3458}, \gamma_{268}, \gamma_{368}, \gamma_{278}, \gamma_{378}, \gamma_{1257}, \gamma_{137}, \gamma_{147}$
(xii)	(IV)	M_{42}	M_{42}	$\gamma_{147}, \gamma_{247}, \gamma_{3457}, \gamma_{267}, \gamma_{367}, \gamma_{148}, \gamma_{248}, \gamma_{3458}, \gamma_{268}, \gamma_{368}$
$(xiii)$	(IV)	M_{42}	M_{46}	$\gamma_{147}, \gamma_{247}, \gamma_{3457}, \gamma_{267}, \gamma_{367}, \gamma_{478}, \gamma_{268}, \gamma_{368}, \gamma_{468}$
(xiv)	(IV)	M_{42}	M_{16}	$\gamma_{147}, \gamma_{247}, \gamma_{3457}, \gamma_{267}, \gamma_{367}, \gamma_{186}, \gamma_{187}, \gamma_{287}, \gamma_{387}$

To complete the proof, we apply Remark 3.2.5 to these cases and show that we end up with one of the varieties varieties $19-37$ in the table of Theorem 3.3.6. Note that the resulting varieties are smooth by Lemma 3.1.6. Applying Remark 3.2.5 to γ_{136} and to γ_{146} yields

$$Q \;=\; \left(\begin{array}{cc|cc|c||ccc} 1 & d_1-1 & 0 & d_1 & d_1/2 & 0 & w_7^1 & w_8^1 \\ 0 & 2 & 1 & 1 & 1 & 0 & w_7^2 & w_8^2 \\ 0 & d_3 & 0 & d_3 & d_3/2 & 1 & w_7^3 & w_8^3 \end{array}\right),$$

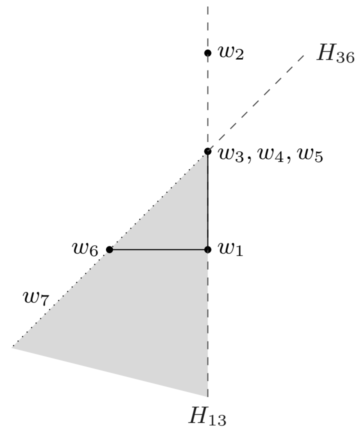
where $d = (d_1, d_2, d_3)$ denotes the degree of g. Note that since $l_{13}(w_2) \le 0$ holds, we have $d_3 \le 0$.

Case (i): We show that this case leads to Nos. 20, 27 and 28 in Theorem 3.3.6.

Remark 3.2.5 applied to γ_{368} and γ_{137} yields $w_8^1 = 1 = w_7^3$. Thus the same remark applied to γ_{147} shows that $d_3 = 0$ or $w_7^2 = 0$ holds. In the first subcase, Remark 3.2.5 together with γ_{1258} yields $w_8^3 = 1$. Applying again Remark 3.2.5, this time to γ_{283} and to γ_{378}, yields $d_1 = 0 = w_7^1$. Thus, Q is given as

$$Q \;=\; \left(\begin{array}{cc|cc|c||ccc} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & w_7^2 & w_8^2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right).$$

The weights are arranged as follows, where w_7 lies somewhere on the dotted line and w_8 somewhere in the gray-shaded area:



We have $Q(\gamma_{283}) \cap Q(\gamma_{136}) \subseteq Q(\gamma_{368})$, which shows that X is of type No. 20, since the semiample cone of X is given by

$$\mathrm{SAmple}(X) \ = \ Q(\gamma_{283}) \ \cap \ Q(\gamma_{137}) \ \cap \ Q(\gamma_{378}) \ \cap \ Q(\gamma_{136}) \, .$$

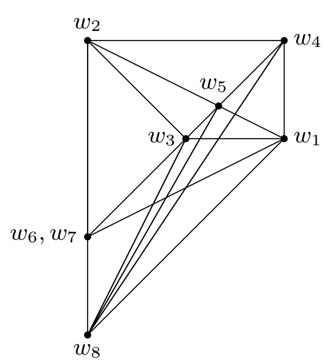
Now we consider the subcase $d_3 \neq 0, w_7^2 = 0$. Remark 3.2.5 together with γ_{468} shows that $d_1 = 0$ or $w_8^2 = 0$ holds. We first treat the possibility $d_3 \neq 0, w_7^2 = 0 = d_1$. Remark 3.2.5 applied to γ_{283} and γ_{284} yields

$$w_8^3 \ = \ 1 - d_3 \qquad \text{and} \qquad -d_3 w_8^2 - d_3 + w_8^3 \ = \ 1 \, .$$

Recall that we are in the case $d_3 \neq 0$. Thus, inserting the first in the second equation gives $w_8^2 = -2$. Now Remark 3.2.5 applied to γ_{378} and to γ_{478} shows that $-w_7^1(d_3+1) = 0 = w_7^1(d_3-1)$ holds. Hence we obtain $w_7^1 = 0$ and Q is given as

$$Q \ = \ \left(\begin{array}{cc|cc|c||ccc} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & 0 & -2 \\ 0 & d_3 & 0 & d_3 & d_3/2 & 1 & 1 & 1-d_3 \end{array}\right), \qquad d_3 < 0 \, .$$

Note that we have $w_3 \in \mathrm{cone}(w_4, w_6)$, $\mathrm{cone}(w_1, w_2) \cap \mathrm{cone}(w_3, w_4) = \mathrm{cone}(w_5)$ as well as $w_2 + w_8 = w_6$. Thus, the weights are arranged as follows:



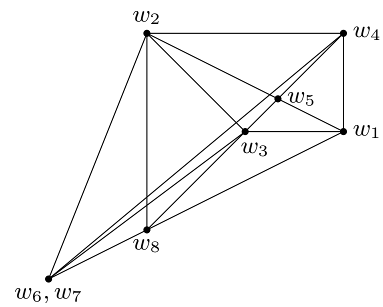
We conclude that X is of type No. 27, since the semiample cone of X is given by

$$\mathrm{SAmple}(X) \ = \ Q(\gamma_{368}) \ \cap \ Q(\gamma_{136}) \, .$$

Now we treat the possibility $d_3 \neq 0, w_7^2 = 0, d_1 \neq 0, w_8^2 = 0$. Remark 3.2.5 applied to γ_{1258}, γ_{378} and γ_{283} yields $w_8^3 = 1, w_7^1 = 0$ and $d_1 = d_3$. Hence, multiplying Q with an unimodular matrix from the left gives

$$Q \ = \ \left(\begin{array}{cc|cc|c||ccc} 1 & -1 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & d_3 & 0 & d_3 & d_3/2 & 1 & 1 & 1 \end{array}\right), \qquad d_3 < 0 \, .$$

Note that we have $w_3 \in \operatorname{cone}(w_4, w_8)$, $\operatorname{cone}(w_1, w_2) \cap \operatorname{cone}(w_3, w_4) = \operatorname{cone}(w_5)$ as well as $w_1 + w_6 = w_8$. Thus, the weights are arranged as follows:



Note that X is of type No. 28, since the semiample cone of X is given by

$$\operatorname{SAmple}(X) \;=\; Q(\gamma_{368}) \;\cap\; Q(\gamma_{283}) .$$

Case (ii): We show that this case leads to Nos. 19 and 26 in Theorem 3.3.6.

Remark 3.2.5 applied to γ_{368} and to γ_{137} yields $w_7^3 = 1 = w_8^1$. Applying again Remark 3.2.5, this time to γ_{468} and to γ_{268}, shows that $w_8^2 = -1$ and $d_1 = 0$ holds. Thus the same remark applied to γ_{147}, $\gamma_{278}, \gamma_{378}$ and to γ_{478} gives

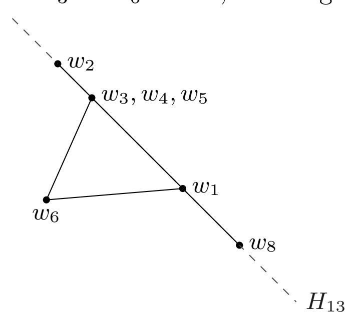
$$d_3 w_7^2 \;=\; 0, \qquad w_7^2 w_8^3 \;=\; 0, \qquad w_7^1 w_8^3 \;=\; 0, \qquad d_3 w_7^1 \;=\; 0 .$$

We obtain the following two cases: $d_3 = 0, w_8^3 = 0$ and $w_7^1 = w_7^2 = 0$.

In the first subcase, we have $d_3 = 0, w_8^3 = 0$. Hence the degree matrix Q is given by

$$Q \;=\; \left(\begin{array}{cc|cc|c||ccc} 1 & -1 & 0 & 0 & 0 & 0 & w_7^1 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & w_7^2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right).$$

Note that we have $w_3 + w_8 = w_1$ and that w_7 lies on the same side of the hypersurface H_{13} through w_1 and w_3 as w_6. Thus, the weights are arranged as follows:

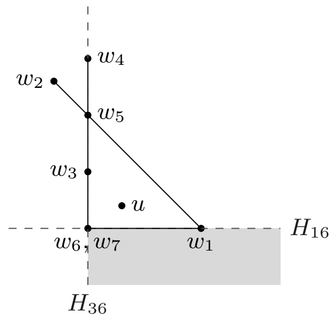


Note that X is of type No. 19, since the semiample cone of X is the intersection of $Q(\gamma_{136})$ and $Q(\gamma_{137})$.

In the second subcase, we have $w_7^1 = w_7^2 = 0$. Hence the degree matrix Q is given by

$$Q \;=\; \left(\begin{array}{cc|cc|c||ccc} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & 0 & -1 \\ 0 & d_3 & 0 & d_3 & d_3/2 & 1 & 1 & w_8^3 \end{array}\right).$$

Note that w_8 lies on the same side of the hypersurface H_{36} through w_3 and w_6 as w_1 and on the opposite side of the hypersurface H_{16} through w_1 and w_6 as w_3. Furthermore, since $d_3 \leq 0$ holds, we have $w_3 \in \operatorname{cone}(w_4, w_6)$. Thus, the weights are arranged as follows, where w_8 lies somewhere in the gray-shaded area:



We conclude that X is of type No. 26, since the semiample cone of X is given by

$$\mathrm{SAmple}(X) \;=\; Q(\gamma_{286}) \;\cap\; Q(\gamma_{137}) \;\cap\; Q(\gamma_{368})\,.$$

Case (iii): We show that this case leads to Nos. 21, 29 and 30 in Theorem 3.3.6.

Remark 3.2.5 applied to γ_{36i} and to γ_{46i}, $i = 7, 8$, yields $w_7^1 = w_8^1 = 1$ and $d_1 w_7^2 = 0 = d_1 w_8^2$. We first treat the subcase $d_1 = 0$. Consider $i \in \{7, 8\}$. Remark 3.2.5 applied to γ_{32i} and to γ_{42i} yields

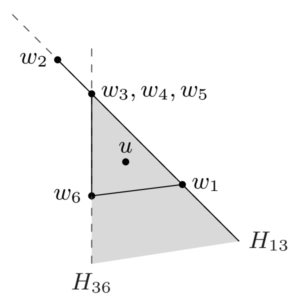
$$w_i^3 \;=\; 1 - d_3 \qquad \text{and} \qquad -\, d_3 w_i^2 - d_3 + w_i^3 \;=\; 1\,.$$

Inserting the first into the second equation gives $d_3(-w_i^2 - 2) = 0$. We conclude that we have $d_3 = 0$ or $w_7^2 = w_8^2 = -2$.

If $d_3 = 0$ holds, Q is given by

$$Q \;=\; \left(\begin{array}{cc|cc|c||ccc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & w_7^2 & w_8^2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right).$$

Note that w_7 and w_8 lie on the same side of the hypersurface H_{13} through w_1 and w_3 as w_6 and on the same side of the hypersurface H_{36} through w_3 and w_6 as w_1. Thus, the weights are arranged as follows, where w_7 and w_8 lie somewhere in the gray-shaded area:



We may assume that $w_8^2 \leq w_7^2$ holds, i.e. we have $w_7 \in \mathrm{cone}(w_3, w_8)$. Thus, the semiample cone of X is given by

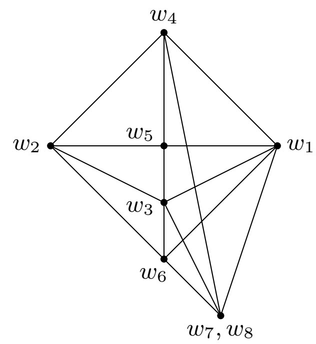
$$\mathrm{SAmple}(X) \;=\; Q(\gamma_{136}) \;\cap\; Q(\gamma_{327}) \;\cap\; Q(\gamma_{367})\,,$$

i.e. X is of type No. 21.

If $w_7^2 = w_8^2 = -2$ holds, Q is given by

$$Q \;=\; \left(\begin{array}{cc|cc|c||ccc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & -2 & -2 \\ 0 & d_3 & 0 & d_3 & d_3/2 & 1 & 1-d_3 & 1-d_3 \end{array}\right).$$

Note that we have $w_3 \in \mathrm{cone}(w_4, w_6)$ as well as $w_2 + w_7 = w_6$. Thus, the weights are arranged as follows, where $w_7 = w_8$ lies somewhere on the dotted line:

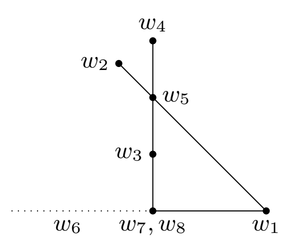


The semiample cone of X is given by $\mathrm{SAmple}(X) = Q(\gamma_{136}) \cap Q(\gamma_{367})$, i.e. X is of type No. 29.

Now we treat the subcase $d_1 \neq 0, w_7^2 = w_8^2 = 0$. Remark 3.2.5 applied to γ_{125i} yields $w_i^3 = 1$, $i = 7, 8$. Again Remark 3.2.5, this time applied to γ_{327}, yields $d_1 = d_3$. Hence, multiplying Q with an unimodular matrix from the left gives

$$Q = \left(\begin{array}{cc|cc|c||ccc} 1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & d_3 & 0 & d_3 & d_3/2 & 1 & 1 & 1 \end{array}\right).$$

Note that we have $w_3 \in \mathrm{cone}(w_4, w_7)$ as well as $w_1 + w_6 = w_7$. Thus the weights are arranged as follows, where w_6 lies somewhere on the dotted line:



The semiample cone of X is given by $\mathrm{SAmple}(X) = Q(\gamma_{327}) \cap Q(\gamma_{367})$, i.e. X is of type No. 30.

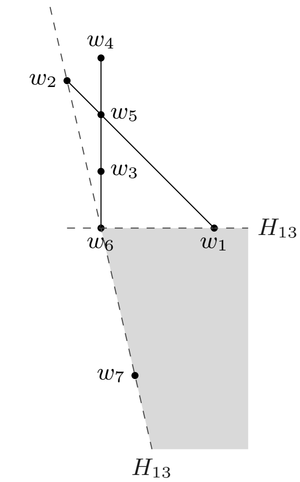
Case (iv): We show that this case leads to Nos. 22 and 31 in Theorem 3.3.6.

Remark 3.2.5 applied to γ_{36i}, $i = 7, 8$, yields $w_7^1 = w_8^1 = 1$. Applying the same remark, this time to γ_{368} and to γ_{468}, shows that $w_8^2 = -1$ and $d_1 = 0$ hold. Now again Remark 3.2.5, this time together with γ_{327} and γ_{427}, implies that $w_7^3 = 1 - d_3$ as well as $0 = d_3(w_7^2 + 2)$ hold. We distinguish the subcases $w_7^2 = -2$ and $w_7^2 \neq -2$.

In the first subcase, we have $w_7^2 = -2$ and $w_7^3 = 1 - d_3$. Hence, Q is given by

$$Q = \left(\begin{array}{cc|cc|c||ccc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & -2 & -1 \\ 0 & d_3 & 0 & d_3 & d_3/2 & 1 & 1 - d_3 & w_8^3 \end{array}\right).$$

Note that we have $w_3 \in \mathrm{cone}(w_4, w_6)$ as well as $w_2 + w_7 = w_6$. Furthermore, w_8 lies on the same side of the hypersurface H_{27} through w_2 and w_7 as w_1 and on the opposite side of the hypersurface H_{16} through w_1 and w_6 as w_3. Thus the weights are arranged as follows, where w_8 lies somewhere in the gray-shaded area:



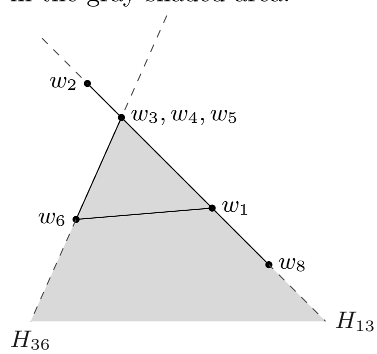
We conclude that X is of type No. 31, since the semiample cone of X is given by

$$\mathrm{SAmple}(X) \;=\; Q(\gamma_{136}) \;\cap\; Q(\gamma_{367}) \;\cap\; Q(\gamma_{368}) \;\cap\; Q(\gamma_{268})\,.$$

In the second subcase, we have $w_7^2 \neq -2$ and thus $d_3 = 0$ and $w_7^3 = 1$ hold. Remark 3.2.5 applied to γ_{278} yields $w_8^3 = 0$. Hence, Q is given by

$$Q \;=\; \left(\begin{array}{cc|cc|c||ccc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & w_7^2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right), \qquad w_7^2 \neq -2\,.$$

Note that we have $w_3 + w_8 = w_1$. Furthermore, w_7 lies on the same side of the hypersurface H_{13} through w_1 and w_3 as w_6 and on the same side of the hypersurface H_{36} through w_3 and w_6 as w_1. Thus the weights are arranged as follows, where w_7 lies somewhere in the gray-shaded area:



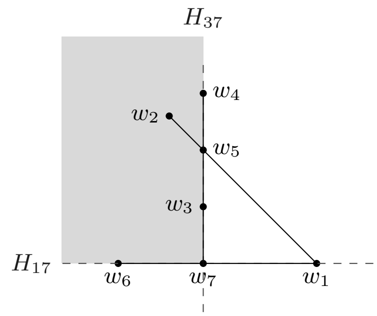
We conclude that X is of type No. 22, since the semiample cone of X is given by the intersection of $Q(\gamma_{136})$ and $Q(\gamma_{327})$.

Case (v): We show that this case leads to Nos. 23, 24 and 32 in Theorem 3.3.6.

Remark 3.2.5 applied to γ_{367}, to γ_{168} and to γ_{467} yields $w_7^1 = 1 = w_8^2$ as well as $d_1 w_7^2 = 0$. We distinguish the subcases $d_1 \neq 0$ and $d_1 = 0$. In the first subcase, we have $w_7^2 = 0$. Remark 3.2.5 together with γ_{187} shows that $w_7^3 = 1$ holds. Thus the same remark together with γ_{327} and γ_{387} yields $d_1 = d_3$ and $w_8^1 = w_8^3 - 1$. Hence, multiplying Q with an unimodular matrix from the left gives

$$Q \;=\; \left(\begin{array}{cc|cc|c||ccc} 1 & -1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 2 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & d_3 & 0 & d_3 & d_3/2 & 1 & 1 & w_8^3 \end{array}\right).$$

Note that we have $w_3 \in \mathrm{cone}(w_4, w_7)$ as well as $w_1 + w_6 = w_7$. Furthermore, w_8 lies on the same side of the hypersurface H_{17} through w_1 and w_7 as w_3 and on the same side of the hypersurface H_{37} through w_3 and w_7 as w_2. Thus the weights are arranged as follows, where w_8 lies somewhere in the gray-shaded area:



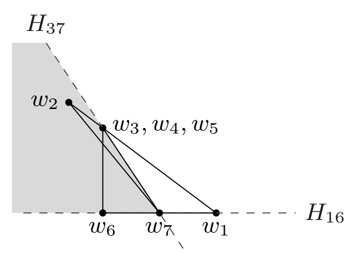
We conclude that X is of type No. 32, since the semiample cone of X is given by the intersection of $Q(\gamma_{237})$, $Q(\gamma_{367})$, $Q(\gamma_{378})$ and $Q(\gamma_{178})$.

In the second subcase, we have $d_1 = 0$ and Remark 3.2.5 together with γ_{327} and γ_{427} yields $w_7^3 = 1 - d_3$ as well as $0 = d_3(w_7^2 + 2)$. The same remark together with γ_{387} and γ_{487} shows that $0 = d_3(1 - w_7^2 w_8^1)$ holds. If $d_3 \neq 0$ held, then we would have $w_7^2 = -2$ and $w_7^2 w_8^1 = 1$, a contradiction. Thus, $d_3 = 0$ and $w_7^3 = 1$ hold. Remark 3.2.5 applied to γ_{187} and to γ_{387} yields $w_7^2 w_8^3 = 0$ as well as $w_8^1 = w_8^3 - 1$. Thus we have the following two possibilities for Q:

The first possibility is that $w_7^2 = 0$ holds, i.e. Q is given by

$$Q = \left(\begin{array}{cc|cc|c||ccc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & w_8^3 - 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & w_8^3 \end{array}\right).$$

Note that we have $w_1 + w_6 = w_7$. Furthermore, w_8 lies on the same side of the hypersurface H_{16} through w_1 and w_6 as w_3 and on the same side of the hypersurface H_{37} through w_3 and w_7 as w_2. Thus the weights are arranged as follows, where w_8 lies somewhere in the gray-shaded area:

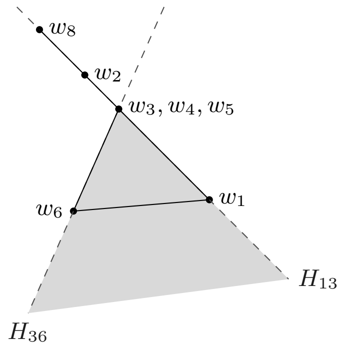


We conclude that X is of type No. 23, since the semiample cone of X is given by the intersection of $Q(\gamma_{237})$, $Q(\gamma_{367})$, $Q(\gamma_{378})$ and $Q(\gamma_{178})$.

The second possibility is that $w_8^3 = 0$ and $w_8^1 = -1$ hold. Here, Q is given by

$$Q = \left(\begin{array}{cc|cc|c||ccc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 2 & 1 & 1 & 1 & 0 & w_7^2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right).$$

Note that we have $w_3 + w_8 = w_2$. Furthermore, w_7 lies on the same side of the hypersurface H_{13} through w_1 and w_3 as w_6 and on the same side of the hypersurface H_{36} through w_3 and w_6 as w_1. Thus the weights are arranged as follows, where w_7 lies somewhere in the gray-shaded area:



We conclude that X is of type No. 24, since the semiample cone of X is given by the intersection of $Q(\gamma_{237})$ and $Q(\gamma_{136})$.

Case (vi): We show that this case leads to Nos. 25 and 33 in Theorem 3.3.6.

Remark 3.2.5 applied to γ_{367} and to γ_{168} yields $w_7^1 = 1 = w_8^2$. The same remark together with γ_{267} and with γ_{467} shows that $w_7^2 = -1$ and $d_1 = 0$ hold. Again Remark 3.2.5, this time applied to γ_{187} and to γ_{387}, yields $w_8^3 = 1 - w_7^3$ as well as $0 = w_7^3(w_8^1 + 1)$ $(*)$. Furthermore, we have

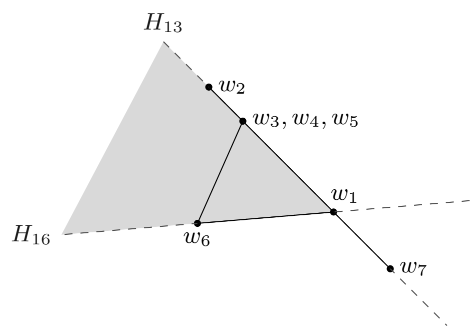
$$1 \;=\; \det(w_2, w_8, w_7) \;=\; 2\det(w_3, w_8, w_7) - \det(w_1, w_8, w_7) - d_3(w_8^1 + 1)\,.$$

We conclude that $d_3(w_8^1 + 1) = 0$ holds. Together with $(*)$, we obtain the two subcases $d_3 = 0 = w_7^3$ and $w_8^1 = -1$.

In the first subcase, Q is given by

$$Q \;=\; \left(\begin{array}{cc|cc|c||ccc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & w_8^1 \\ 0 & 2 & 1 & 1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right).$$

Note that we have $w_3 + w_7 = w_1$. Furthermore, w_8 lies on the same side of the hypersurface H_{13} through w_1 and w_3 as w_6 and on the same side of the hypersurface H_{16} through w_1 and w_6 as w_3. Thus the weights are arranged as follows, where w_8 lies somewhere in the gray-shaded area:

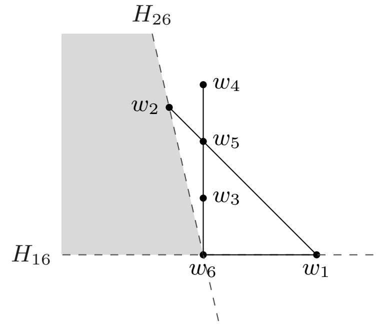


We have $Q(\gamma_{136}) \cap Q(\gamma_{178}) \subseteq Q(\gamma_{186})$, which shows that the semiample cone of X is given by $Q(\gamma_{136}) \cap Q(\gamma_{178})$, i.e. X is of type No. 25.

In the second subcase, Q is given by

$$Q \;=\; \left(\begin{array}{cc|cc|c||ccc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 2 & 1 & 1 & 1 & 0 & -1 & 1 \\ 0 & d_3 & 0 & d_3 & d_3/2 & 1 & w_7^3 & 1 - w_7^3 \end{array}\right).$$

Note that we have $w_3 \in \operatorname{cone}(w_4, w_6)$ as well as $w_7 + w_8 = w_6$. Furthermore, w_8 lies on the same side of the hypersurface H_{16} through w_1 and w_6 as w_3 and on the opposite side of the hypersurface H_{26} through w_2 and w_6 as w_1. Thus the weights are arranged as follows, where w_8 lies somewhere in the gray-shaded area:



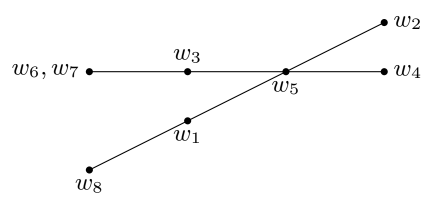
Since we have $Q(\gamma_{267}) \subseteq Q(\gamma_{278}) \cap Q(\gamma_{136})$, we conclude that the semiample cone of X is given by the intersection of $Q(\gamma_{136})$, $Q(\gamma_{367})$, $Q(\gamma_{168})$ and $Q(\gamma_{278})$. Hence, X is of type No. 33.

Case (vii): We show that this case leads to No. 34 in Theorems 3.3.6.

Remark 3.2.5 applied to γ_{137} and to γ_{138} yields $w_7^3 = 1 = w_8^3$. The same remark together with γ_{268} and γ_{468} yields $w_8^1 = -w_8^2$ as well as $1 = -w_8^2(d_1+1)$. Thus we obtain $d_1 = 0, w_8^2 = -1$ or $d_1 = -2, w_8^2 = 1$. If $d_1 = -2, w_8^2 = 1$ held, then Remark 3.2.5 applied to γ_{238} would yield $d_3 = 4$, contradicting $d_3 \leq 0$. Hence we obtain $d_1 = 0, w_8^2 = -1$. Remark 3.2.5 applied to γ_{238}, γ_{147} and to γ_{478} yields $d_3 = -2$, $w_7^2 = 0$ as well as $w_7^1 = 0$. Hence, Q is given by

$$Q \;=\; \left(\begin{array}{cc|cc|c||ccc} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & 0 & -1 \\ 0 & -2 & 0 & -2 & -1 & 1 & 1 & 1 \end{array}\right).$$

Note that the weights are arranged as follows:



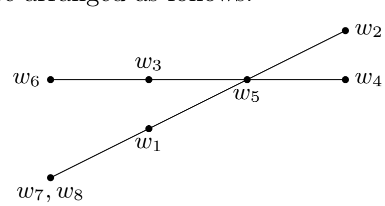
We conclude that X is of type No. 34, since the semiample cone of X is given by the intersection of $Q(\gamma_{136})$ and $Q(\gamma_{138})$.

Case (viii): We show that this case leads to No. 35 in Theorem 3.3.6.

Applying Remark 3.2.5 to $\gamma_{138}, \gamma_{268}, \gamma_{468}$ and to γ_{238} as in case (vii), we obtain $d_1 = 0$ and $w_8 = (1,-1,1)$. Analogously we conclude that $w_7 = (1,-1,1)$ holds. Furthermore, Remark 3.2.5 applied to γ_{238} yields $d_3 = -2$. Hence, Q is given by

$$Q \;=\; \left(\begin{array}{cc|cc|c||ccc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & -1 & -1 \\ 0 & -2 & 0 & -2 & -1 & 1 & 1 & 1 \end{array}\right).$$

Note that the weights are arranged as follows:



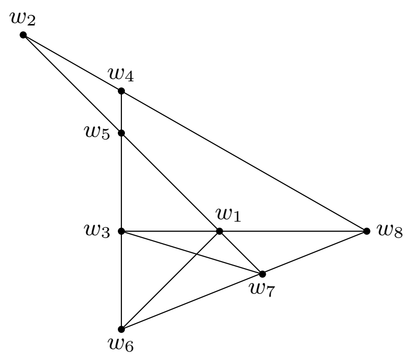
We conclude that X is of type No. 35, since the semiample cone of X is given by the intersection of $Q(\gamma_{136})$ and $Q(\gamma_{138})$.

Case (ix): We show that this case leads to No. 36 in Theorem 3.3.6.

As in case (viii), we obtain $w_7 = (1, -1, 1)$ and $d_1 = 0$. Applying Remark 3.2.5 to $\gamma_{368}, \gamma_{237}, \gamma_{268}$ and to γ_{378} shows that $w_8^1 = 1, d_3 = -2, w_8^2 = -1$ and $w_{83} = 0$ hold. Hence, Q is given by

$$Q = \left(\begin{array}{cc|cc|c||ccc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & -1 & -1 \\ 0 & -2 & 0 & -2 & -1 & 1 & 1 & 0 \end{array}\right).$$

Note that we have $2w_6 + w_4 = w_3$, $w_7 + w_5 = w_1$, $w_8 + w_6 = w_7$, $w_8 + w_3 = w_1$ as well as $w_2 + w_8 = w_4$. Hence the weights are arranged as follows:



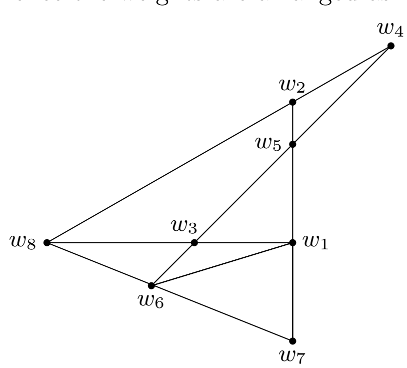
We conclude that X is of type No. 36, since the semiample cone of X is given by the intersection of $Q(\gamma_{136})$ and $Q(\gamma_{137})$.

Case (x): We show that this case leads to No. 37 in Theorem 3.3.6.

As in case (viii), we obtain $w_7 = (1, -1, 1)$ and $d_1 = 0$. Applying Remark 3.2.5 to γ_{186}, γ_{187}, γ_{237} and to γ_{487}, shows that $w_8^2 = 1$, $w_8^3 = 0$, $d_3 = -2$ and $w_8^1 = -1$ hold. Hence, Q is given by

$$Q = \left(\begin{array}{cc|cc|c||ccc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 2 & 1 & 1 & 1 & 0 & -1 & 1 \\ 0 & -2 & 0 & -2 & -1 & 1 & 1 & 0 \end{array}\right).$$

Note that we have $2w_6 + w_4 = w_3$, $w_7 + w_5 = w_1$, $w_8 + w_7 = w_6$, $w_8 + w_1 = w_3$ as well as $w_4 + w_8 = w_2$. Hence the weights are arranged as follows:



We conclude that X is of type No. 37, since the semiample cone of X is given by the intersection of $Q(\gamma_{136})$ and $Q(\gamma_{137})$.

Cases (xi)-(xiv): We show that there are no smooth varieties in these cases.

Note that in all these cases there is $7 \le \ell \le 8$ such that $\gamma_{14\ell}, \gamma_{24\ell}, \gamma_{26\ell}$ and $\gamma_{36\ell}$ are relevant faces. Applying Remark 3.2.5 to $\gamma_{36\ell}$ and $\gamma_{26\ell}$ shows $w_\ell^2(1 - d_1) = -1$ and $w_\ell^1 = 1$. We conclude that we have either $w_\ell^2 = -1$, $d_1 = 0$ or $w_\ell^2 = 1$, $d_1 = 2$.

If $w_\ell^2 = -1$ and $d_1 = 0$ held, Remark 3.2.5 together with $\gamma_{24\ell}$ and $\gamma_{14\ell}$ would yield $w_\ell^3 = -1$ and $d_3 = 2$, contradicting $d_3 \leq 0$. Thus we obtain $w_\ell^2 = 1$ and $d_1 = 2$. Now again Remark 3.2.5, applied this time to $\gamma_{14\ell}$ and to $\gamma_{24\ell}$, yields $d_3 = -4$ and $w_\ell^3 = -3$. But since $Q(\gamma_{136}) \subseteq \mathbb{Q}^3_{\geq 0}$ as well as $w_1^3 = 0$ holds, w_ℓ^3, $w_4^3 = d_3 < 0$ contradict $Q(\gamma_{136})^\circ \cap Q(\gamma_{14\ell})^\circ \neq \emptyset$. Hence there are no smooth varieties in these cases. $\square$

CHAPTER 4

Base point free questions

This chapter investigates the *base point free monoid*, i.e. the monoid of base point free Cartier divisor classes of a Mori dream space, and also concerns Fujita's base point free conjecture. Part of this chapter, namely Section 4.1 and Sections 4.7 – 4.9, have been presented in [**26**].

Section 4.1 deals with *embedded monoids*, that means finitely generated monoids in finitely generated abelian groups, and thereby generalizes ideas of the theory on affine semigroups [**18**, Chapter 2] to monoids with non-trivial torsion part. In the subsequent sections, to be precise in Sections 4.2 – 4.4, we investigate the base point free monoid, first of a toric variety, then of a Mori dream space and finally of a variety with a torus action of complexity one. It is well-known that for Cartier divisor classes on complete toric varieties, semiampleness implies base point freeness, i.e. in this case, the base point free monoid is saturated. For smooth rational projective varieties with a torus action of complexity one and Picard number two, the same statement follows from the classification done in Chapter two, but the assertion is no longer true if we consider locally factorial varieties of Picard number two or smooth varieties of arbitrary Picard number, see Example 4.5.1 and Example 4.8.4, respectively. In this chapter, we give some criteria for the base point free monoid to be saturated. As a further base point free question, we study Fujita's base point free conjecture. Recall that in the end of the eighties, Takao Fujita conjectured the following:

Conjecture 4.0.1. (Fujita's base point free conjecture [32]) *Let X be a smooth projective variety with canonical class $\mathcal{K}_X$. Then $\mathcal{K}_X + m\mathcal{L}$ is base point free for all $m \geq \dim(X) + 1$ and for all ample Cartier divisor classes $\mathcal{L}$.*

We prove some sufficient criteria for a variety to fulfill Conjecture 4.0.1 in Sections 4.2 – 4.4. As an application of the classifications done in Chapters two and three, we furthermore provide sample classes of varieties fulfilling Fujita's base point free conjecture, see Corollaries 4.3.9 and 4.4.14.

Note that for varieties $X = X(R, \mathfrak{F}, \Phi)$ arising from a bunched ring, Conjecture 4.0.1 is a question of the study of monoids: It is sufficient to show that $\mathcal{K}_X + (\dim(X) + 1)\mathcal{L}$ is an element of the conductor ideal of the base point free monoid for all ample Cartier divisor classes $\mathcal{L}$. With this in mind, in Sections 4.5 and 4.6, we investigate Fujita's base point free conjecture for singular rational varieties with a torus action of complexity one and Picard number at most two using Frobenius numbers and their generalization to higher dimensions.

In Sections 4.7 – 4.9 that form the final part of the chapter, we present and prove algorithms concerning embedded monoids and base point free questions of Mori dream spaces. Section 4.7 deals with algorithms for embedded monoids, among others for computing generators of intersections of embedded monoids and for computing an element of the conductor ideal; see Algorithms 4.7.1, 4.7.3, 4.7.5 and 4.7.7. In Section 4.8, we apply these algorithms to base point free questions of Mori dream spaces. Section 4.9 contains our main algorithm, Algorithm 4.9.4, which tests Fujita's base point free conjecture for $\mathbb{Q}$-factorial Mori dream spaces with known canonical class $\mathcal{K}_X$. The latter is quite often the case; for instance if X is spherical or if its Cox ring is a complete intersection, see Remark 4.9.1 for details.

In **[27]**, we provide an implementation of our algorithms building on the two `Maple`-based software packages `convex` **[29]** and `MDSpackage` **[38]**. Using this implementation, we present a first example of a smooth $\mathbb{K}^*$-surface having a semiample Cartier divisor with base points, see Example 4.8.4, and we prove Fujita's base point free conjecture for a six-dimensional Mori dream space, see Example 4.9.5.

4.1. Embedded monoids

This section concerns numerical monoids and their generalizations to monoids in higher dimensions having possibly non-trivial torsion part. We present assertions such as Lemma 4.1.13 and Proposition 4.1.15 concerning the intersection of monoids. We will need these statements later on when investigating the base point free monoid of Mori dream spaces. Moreover, we provide lemmata concerning the Frobenius number and the conductor ideal which will be crucial for the investigation of Fujita's base point free conjecture for varieties with a torus action of complexity one and Picard number one and two, see Sections 4.5 and 4.6.

A monoid $S \subseteq \mathbb{N}$ is called a *numerical monoid* if $\mathrm{lin}_{\mathbb{Z}}(S) = \mathbb{Z}$ holds. Note that a monoid $S = \mathrm{lin}_{\mathbb{Z}_{\geq 0}}(w_1, \ldots, w_r)$, $w_i \in \mathbb{Z}_{\geq 1}$, is a numerical monoid if and only if the integers $w_1, \ldots, w_r$ are coprime. For a numerical monoid $S = \mathrm{lin}_{\mathbb{Z}_{\geq 0}}(w_1, \ldots, w_r)$ generated by $w_i \in \mathbb{Z}_{\geq 1}$, the *Frobenius number* $\mathcal{F}(S) = \mathcal{F}(w_1, \ldots, w_r)$ is the least integer $x \in \mathbb{Z}$ such that $x + n \in S$ holds for all $n \in \mathbb{Z}_{\geq 1}$. In this case, $x + 1$ is called *conductor* of S. The Frobenius problem, i.e. the problem of finding the Frobenius number, has attracted substantial attention, see, for instance, **[5, 62]**. For $r = 2$ one can use Sylvester's formula to compute the Frobenius number.

Proposition 4.1.1. (Sylvester's formula [66]) *We have $\mathcal{F}(w_1, w_2) = w_1 w_2 - w_1 - w_2$ for any two coprime integers $w_1, w_2 \in \mathbb{Z}_{\geq 0}$.*

For $r \geq 3$, it is in some sense not possible to determine the Frobenius number via a formula. Indeed, Curtis **[22]** proved the following:

Proposition 4.1.2. [22] *There is no finite set $\{f_1, \ldots, f_n\}$ of polynomials such that for each choice of integers $w_1, w_2, w_3 \in \mathbb{Z}_{\geq 1}$ whose greatest common divisor is one, there is some $1 \leq i \leq n$ such that $f_i(w_1, w_2, w_3) = \mathcal{F}(w_1, w_2, w_3)$ holds.*

Nevertheless, there are many formulas for special cases of $\mathcal{F}(w_1, \ldots, w_r)$ as well as upper and lower bounds. For a comprehensive overview see **[5]**. Here we give some formulas for computing the Frobenius number which we will need later on.

Lemma 4.1.3. [15] *Let $w_1, \ldots, w_r \in \mathbb{Z}_{\geq 1}$ be integers whose greatest common divisor is one and set $d := gcd(w_1, \ldots, w_{r-1})$. Then the following holds for the Frobenius number of the numerical monoid $S = \mathrm{lin}_{\mathbb{Z}}(w_1, \ldots, w_r)$:*

$$\mathcal{F}(w_1, \ldots, w_r) \;=\; d\,\mathcal{F}\Big(\frac{w_1}{d}, \ldots, \frac{w_{r-1}}{d}, w_r\Big) + (d-1)w_r\,.$$

Lemma 4.1.4. *Let $l_1, \ldots, l_r \in \mathbb{Z}_{\geq 1}$ be integers whose greatest common divisor is one and set $w_\ell := l_1 \cdots l_{\ell-1} l_{\ell+1} \cdots l_r$. Then the following holds for the Frobenius number of the numerical monoid $S = \mathrm{lin}_{\mathbb{Z}}(w_1, \ldots, w_r)$:*

$$\mathcal{F}(w_1, \ldots, w_r) \quad = \quad (r-1)\prod_{i=1}^{r} l_i \;-\; \sum_{\ell=1}^{r} w_\ell\,.$$

Proof. We proceed by induction on r. For $r = 1$, $l_1 = 1 = w_1$ holds. We thus obtain $\mathcal{F}(w_1) = -1$, which coincides with the formula one the right-hand side. For

$r = 2$, the above formula equals Sylvester's formula. Now assume that the statement is true for $\mathcal{F}(w_1, \ldots, w_{r-1})$. We obtain

$$\begin{aligned} \mathcal{F}(w_1, \ldots, w_r) &= l_r\, \mathcal{F}\left(\prod_{\substack{i=1 \\ i \neq 1}}^{r-1} l_i, \, \ldots\, , \prod_{\substack{i=1 \\ i \neq r-1}}^{r-1} l_i, \; w_r \right) \; + \; (l_r - 1)\, w_r \\ &= l_r \left((r-2) \prod_{i=1}^{r-1} l_i \; - \; \sum_{\ell=1}^{r-1} \prod_{\substack{i=1 \\ i \neq \ell}}^{r-1} l_i \right) \; + \; (l_r - 1)\, w_r \\ &= (r-1) \prod_{i=1}^{r} l_i \; - \; \sum_{\ell=1}^{r} w_\ell \, , \end{aligned}$$

where the first equality holds according to Lemma 4.1.3 and in the second step we may apply the induction hypothesis since w_r is a multiple of $l_1 \cdots l_{\ell-1} l_{\ell+1} \cdots l_{r-1}$. $\square$

In the following, we consider monoids in arbitrary finitely generated abelian groups and generalize concepts presented in [**18**, Chapter 2] to this case. Let K be a finitely generated abelian group. We denote by $K = K^0 \oplus K^{\mathrm{tor}}$ the decomposition of K into free and torsion part and we write $K_{\mathbb{Q}} := K \otimes_{\mathbb{Z}} \mathbb{Q}$ for the associated rational vector space. Note that each $w \in K = K^0 \oplus K^{\mathrm{tor}}$ can be represented as $w = (w^0, w^{\mathrm{tor}})$ with unique elements $w^0 \in K^0$ and $w^{\mathrm{tor}} \in K^{\mathrm{tor}}$. Every $w \in K$ defines an element $w \otimes 1 \in K_{\mathbb{Q}}$, which we denote as well by w for short. A *cone* in a rational vector space always refers to a convex, polyhedral cone. The relative interior of a cone $\tau \subseteq K_{\mathbb{Q}}$ is denoted by τ°.

By an *embedded monoid* we mean a pair $S \subseteq K$, where S is a finitely generated submonoid of K. For an embedded monoid $S \subseteq K$, we denote by

$$\mathrm{cone}(S) \; := \; \mathrm{cone}(w \otimes 1; \; w \in S) \; \subseteq \; K_{\mathbb{Q}}$$

the (convex, polyhedral) cone generated by the elements of S. An embedded monoid $S \subseteq K$ is *spanning* if S generates K as a group, i.e. if $\mathrm{lin}_{\mathbb{Z}}(S) = K$ holds. In particular, numerical monoids are spanning embedded monoids $S \subseteq \mathbb{Z}$. The *saturation* of an embedded monoid $S \subseteq K$ is the embedded monoid

$$\tilde{S} \; := \; \{w \in K; \; nw \in S \text{ for some } n \in \mathbb{Z}_{\geq 1}\} \; \subseteq \; K \, .$$

An embedded monoid $S \subseteq K$ is called *saturated* if $S = \tilde{S}$ holds. Note that the saturation $\tilde{S} \subseteq K$ of $S \subseteq K$ consists of all $w \in K$ defining an element in $\mathrm{cone}(S) \subseteq K_{\mathbb{Q}}$, i.e. we have the following:

Remark 4.1.5. Let $S \subseteq K$ be an embedded monoid. The saturation of S is given as

$$\tilde{S} \; = \; \iota^{-1}(\mathrm{cone}(S)) \; = \; \iota_0^{-1}\left(\mathrm{cone}\left(x^0 \otimes 1; \; x \in S\right)\right) \times K^{\mathrm{tor}},$$

where ι and ι_0 are the maps $K \to K \otimes \mathbb{Q}$ and $K^0 \to K^0 \otimes \mathbb{Q}$ defined through $w \mapsto w \otimes 1$.

Proof. Let $C := \mathrm{cone}\left(x^0 \otimes 1; \; x \in S\right)$. We will show that $\tilde{S} = \iota_0^{-1}(C) \times K^{\mathrm{tor}}$ holds. For the first inclusion, let $s = (s^0, s^{\mathrm{tor}}) \in \tilde{S}$, i.e. s is an element of K and there is an integer $\alpha \in \mathbb{Z}_{\geq 1}$ s.t. $\alpha s \in S$ holds. This yields $s^{\mathrm{tor}} \in K^{\mathrm{tor}}$ and $\alpha s^0 \otimes 1 \in C$. Since C is convex, we conclude that $s^0 \otimes 1$ is contained in C, i.e. we showed that s^0 is contained in $\iota_0^{-1}(C)$. For the opposite inclusion let $s = (s^0, s^{\mathrm{tor}}) \in \iota_0^{-1}(C) \times K^{\mathrm{tor}}$, i.e. $s^0 = \sum_{i=1}^r \alpha_i x_i^0$ holds with some $\alpha_i \in \mathbb{Q}_{\geq 0}$ and $x_i \in S$. Since s^{tor} as well as the x_i^{tor} are elements of K^{tor}, there are $m \in \mathbb{Z}_{\geq 1}$ and $n_i \in \mathbb{Z}_{\geq 1}$ such that $ms^{\mathrm{tor}} = n_i x_i^{\mathrm{tor}} = 0_{K^{\mathrm{tor}}}$ holds. Set $n := \mathrm{lcm}(m, n_1, \ldots, n_r)$ and denote by d the common denominator of the α_i. Then we have

$$\begin{aligned} dns &= dn\left((s^0, 0_{K^{\mathrm{tor}}}) + (0_{K^0}, s^{\mathrm{tor}})\right) \\ &= dn\left(\sum_{i=1}^r \alpha_i x_i^0,\ 0_{K^{\mathrm{tor}}}\right) + dn\left(0_{K^0}, s^{\mathrm{tor}}\right) \\ &= \left(n\sum_{i=1}^r d\alpha_i x_i^0,\ \sum_{i=1}^r d\alpha_i (n x_i^{\mathrm{tor}})\right) + d\left(n0_{K^0}, ns^{\mathrm{tor}}\right) \\ &= n\sum_{i=1}^r d\alpha_i x_i + 0_K\,. \end{aligned}$$

Note that the $d\alpha_i$ are integers and that x_i and 0_K are elements of S. Hence the last line and thus also dns is contained in S. This shows that $s \in \tilde{S}$ holds. ☐

Lemma 4.1.6. *Let $S \subseteq K$ be an embedded monoid. If S is generated by $\dim(S)$ elements, then $S \subseteq \mathrm{lin}_{\mathbb{Z}}(S)$ is a saturated embedded monoid.*

Proof. Let $w \in \mathrm{lin}_{\mathbb{Z}}(S)$ and $n, r \in \mathbb{Z}_{\geq 1}$ such that $nw \in S$ and $r = \dim(S)$ hold. Consider generators $s_1, \ldots, s_r \in S$ for S. Then there are $a_i \in \mathbb{Z}$ and $b_i \in \mathbb{Z}_{\geq 0}$ such that $w = \sum_{i=1}^r a_i s_i$ and $nw = \sum_{i=1}^r b_i s_i$ holds. In particular, this gives

$$\sum_{i=1}^r a_i s_i^0 \ = \ w^0 \ = \ \sum_{i=1}^r \frac{b_i}{n} s_i^0\,.$$

Note that $(s_1^0, \ldots, s_r^0)$ is a linearly independent family over $\mathbb{Q}$ since $r = \dim(S)$ holds. Thus, we conclude that $a_i = b_i/n$ holds for $i = 1, \ldots, r$. In particular, the integers a_i are greater than or equal to zero, which means that w is contained in S. Thus, S is saturated. ☐

Remark 4.1.7. Let $F\colon K \to K'$ be a homomorphism of finitely generated abelian groups.

(i) If $S \subseteq K$ is a spanning embedded monoid, then $F(S) \subseteq F(K)$ is so.
(ii) If $S' \subseteq K'$ is a spanning embedded monoid, then $F(S')^{-1} \subseteq K$ is so.

Let $S \subseteq K$ be an embedded monoid. A non-empty set $M \subseteq K$ is called an S-*module* if $S + M \subseteq M$ holds. We call an S-module M an *ideal* if $M \subseteq S$ holds and *finitely generated* if there is a finite subset $\{m_1, \ldots, m_\ell\} \subseteq M$ with the property that $M = \{s + m_1, \ldots, s + m_\ell;\ s \in S\}$ holds.

Lemma 4.1.8. *Let $S \subseteq K$ be an embedded monoid. Consider $x_1, \ldots, x_r \in S$ such that $\{x_1 \otimes 1, \ldots, x_r \otimes 1\}$ is a set of generators for $\mathrm{cone}(S)$. Then the finite set*

$$M \ := \ \iota^{-1}\left(\left\{\sum_{i=1}^r \alpha_i (x_i \otimes 1);\ \ \alpha_i \in \mathbb{Q},\ 0 \leq \alpha_i \leq 1\right\}\right),$$

where ι is the map $\iota\colon K \to K \otimes \mathbb{Q},\ w \mapsto w \otimes 1$, generates $\tilde{S}$ as an S-module. In particular, $\tilde{S}$ is a finitely generated S-module.

Proof. By Remark 4.1.5, $\tilde{S}$ is an S-module. In case of a torsion-free group K, the statement on finite generation of $\tilde{S}$ as an S-module is Gordan's Lemma [**21**, Prop. 1.2.17]. The proof extends to the case of finitely generated abelian groups as follows: Denote by ι_0 the map $K^0 \to K^0 \otimes \mathbb{Q},\ w \mapsto w \otimes 1$. Since the set

$$M_0 := \iota_0^{-1}\left(\left\{\sum_{i=1}^r \alpha_i x_i^0;\ \ \alpha_i \in \mathbb{Q},\ 0 \leq \alpha_i \leq 1\right\}\right)$$

is bounded and since K^{tor} is a finite group, $M = M_0 \times K^{\mathrm{tor}}$ is a finite set, say $M = \{m_1, \ldots, m_\ell\}$ with certain elements $m_i \in \tilde{S}$. We claim that

$$\tilde{S} \;=\; \bigcup_{i=1}^{\ell} (m_i + S)$$

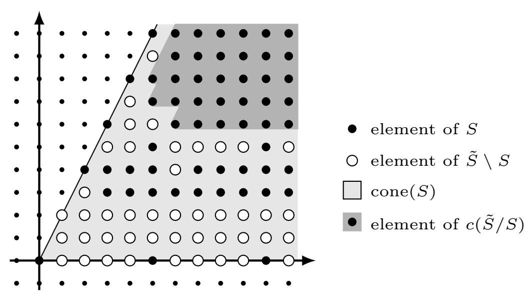
holds, i.e. that M generates the S-module $\tilde{S}$. With Remark 4.1.5, the inclusion "$\supseteq$" is obvious. For the other inclusion, let $y \in \tilde{S}$. We have $y = (y^0, y^{\mathrm{tor}})$, where $y^0 \in \iota_0^{-1}(\mathrm{cone}(x^0 \otimes 1;\ x \in S)) \cap K^0$ and $y^{\mathrm{tor}} \in K^{\mathrm{tor}}$ hold. Pick $\alpha_i \in \mathbb{Q}_{\geq 0}$ such that $y^0 = \sum_{i=1}^r \alpha_i x_i^0$ holds. We obtain

$$\begin{aligned} y &= \Big(\sum_{i=1}^r \alpha_i x_i^0,\ \ y^{\mathrm{tor}}\Big) \\ &= \Big(\sum_{i=1}^r \lfloor\alpha_i\rfloor x_i^0 + \sum_{i=1}^r (\alpha_i - \lfloor\alpha_i\rfloor) x_i^0,\ \ y^{\mathrm{tor}}\Big) \\ &= \sum_{i=1}^r \lfloor\alpha_i\rfloor x_i + \Big(\sum_{i=1}^r (\alpha_i - \lfloor\alpha_i\rfloor) x_i^0,\ \ y^{\mathrm{tor}} - \sum_{i=1}^r \lfloor\alpha_i\rfloor x_i^{\mathrm{tor}}\Big), \end{aligned}$$

where y and the first summand in the bottommost line are contained in K. Thus the same holds for the second summand in the bottommost line. Furthermore, we have $0 \leq (\alpha_i - \lfloor\alpha_i\rfloor) \leq 1$, i.e. the second summand in the bottommost line is one of the m_i's, say m_{i_0}. Note that $\sum_{i=1}^r \lfloor\alpha_i\rfloor x_i$ is an element of S. Thus we showed that $y \in S + m_{i_0}$ holds, which completes the proof. $\square$

Definition 4.1.9. Let $S \subseteq K$ be an embedded monoid. The *conductor ideal* of $S \subseteq K$ is the subset

$$c(\tilde{S}/S) \;:=\; \{x \in S;\ x + \tilde{S} \subseteq S\} \;\subseteq\; S\,.$$



Proposition 4.1.10. *Let $S \subseteq K$ be an embedded monoid. If $S \subseteq K$ is spanning, then the conductor ideal $c(\tilde{S}/S)$ is non-empty, i.e. it is in particular an S-module.*

Proof. By definition, $S + c(\tilde{S}/S) \subseteq c(\tilde{S}/S)$ holds, i.e. we only have to show that $c(\tilde{S}/S)$ is non-empty. In case of a torsion-free group K, one can find a proof in [**18**, Prop. 2.33]. For finitely generated abelian groups we may extend the proof as follows: According to Lemma 4.1.8, we have $\tilde{S} = \{m_1 + s, \ldots, m_\ell + s;\ s \in S\}$ with some finite subset $\{m_1, \ldots, m_\ell\} \subseteq \tilde{S}$. By assumption, the embedded monoid $S \subseteq K$ is spanning. This yields representations $m_i = x_i - y_i$ with $x_i, y_i \in S$. We claim that $z := \sum_{i=1}^{\ell} y_i$ is contained in the conductor ideal $c(\tilde{S}/S)$. Indeed

$$z + m_j = \sum_{\substack{1 \leq i \leq \ell \\ i \neq j}} y_i + x_j \;\in\; S$$

holds for all $1 \leq j \leq \ell$, i.e. we have $z + \tilde{S} \subseteq S$. $\square$

Corollary 4.1.11. *Let $S \subseteq K$ be a spanning embedded monoid and let M be defined as in Lemma 4.1.8. Then the following are equivalent for $w \in K$:*

(i) *The conductor ideal $c(\tilde{S}/S)$ contains w.*
(ii) *For all $m \in M$, $w + m$ is contained in S.*

Lemma 4.1.12. *Let $S \subseteq K$ be a spanning embedded monoid and consider an ideal $S_0 \subseteq S$. If $S \subseteq K$ is spanning, then the same holds for $S_0 \subseteq K$. In particular, $c(\tilde{S}/S) \subseteq K$ then is a spanning embedded monoid.*

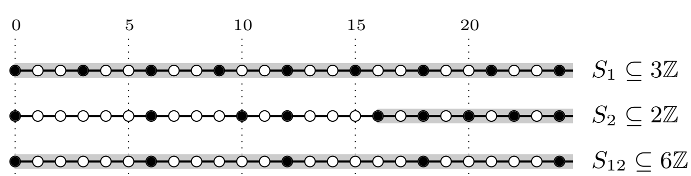
Proof. We show that $\operatorname{lin}_{\mathbb{Z}}(S_0) \supseteq K$ holds if $S \subseteq K$ is spanning. Let $x \in S_0$. Since S_0 is an S-module, $S = -x + x + S$ is contained in $\operatorname{lin}_{\mathbb{Z}}(S_0)$. In particular, we have $\operatorname{lin}_{\mathbb{Z}}(S_0) \supseteq \operatorname{lin}_{\mathbb{Z}}(S) = K$. The supplement is due to Proposition 4.1.10. $\square$

Lemma 4.1.13. *Let K be a finitely generated abelian group and consider two subgroups $K_1, K_2 \subseteq K$. Let $S_i \subseteq K_i$ be embedded monoids with saturations $\tilde{S}_i$. Then the following holds for the intersection $S_{12} := S_1 \cap S_2$:*

(i) *The intersection $S_{12} \subseteq K_1 \cap K_2$ is an embedded monoid.*
(ii) *We have $\tilde{S}_{12} = \tilde{S}_1 \cap \tilde{S}_2$, where $\tilde{S}_{12}$ denotes the saturation of the embedded monoid $S_{12} \subseteq K_1 \cap K_2$.*
(iii) *We have $c(\tilde{S}_1/S_1) \cap c(\tilde{S}_2/S_2) \subseteq c(\tilde{S}_{12}/S_{12})$.*

Proof. For (i), only the finite generation of $S_1 \cap S_2$ needs some explanation, see for instance [**3**, Prop. 1.1.2.2]. To prove the first inclusion of (ii), let $x \in \tilde{S}_{12}$. This means that we have $x \in K_1 \cap K_2$ and that there is $n \in \mathbb{Z}_{\geq 1}$ such that $nx \in S_1 \cap S_2$ holds. Clearly, this shows $x \in \tilde{S}_1 \cap \tilde{S}_2$. To prove the second inclusion, let $x \in \tilde{S}_1 \cap \tilde{S}_2$. Hence $x \in K_1 \cap K_2$ holds and there are $n_1,\, n_2 \in \mathbb{Z}_{\geq 1}$ such that $n_i x \in S_i$, $i = 1, 2$ hold. This means that $n_1 n_2 x$ is contained in $S_1 \cap S_2$, i.e. we have $x \in \tilde{S}_{12}$. For (iii), consider an element $x \in c(\tilde{S}_1/S_1) \cap c(\tilde{S}_2/S_2)$. This means that x is contained in the intersection S_{12} and $x + \tilde{S}_i \subseteq S_i$ holds. With (ii), we conclude that $x + \tilde{S}_{12}$ is contained in S_{12}, i.e. the conductor ideal of $S_{12} \subseteq \operatorname{lin}_{\mathbb{Z}}(S_{12})$ contains x. $\square$

Example 4.1.14. Assertion (iii) of Lemma 4.1.13 is in general a proper inclusion: Consider the embedded monoids $S_1 := 3\mathbb{Z}_{\geq 0} \subseteq K_1 := 3\mathbb{Z}$ and $S_2 := \operatorname{lin}_{\mathbb{Z}_{\geq 0}}(6, 10) \subseteq K_2 := 2\mathbb{Z}$. Then the situation is as follows,



where the gray-shaded area indicates the conductor ideals of the monoids S_1, S_2 and S_{12}, i.e. we have $c(\tilde{S}_1/S_1) = 3\mathbb{Z}_{\geq 0}$, $c(\tilde{S}_2/S_2) = 16 + 2\mathbb{Z}_{\geq 0}$ and $c(\tilde{S}_{12}/S_{12}) = 6\mathbb{Z}_{\geq 0}$. Note that the latter is a proper superset of

$$c(\tilde{S}_1/S_1) \ \cap \ c(\tilde{S}_2/S_2) \ = \ 18 + 6\mathbb{Z}_{\geq 0}\,.$$

Proposition 4.1.15. *Let K_1 and K_2 be subgroups of a finitely generated abelian group K and consider embedded monoids $S_i \subseteq K_i$, $i = 1, 2$. If $\operatorname{cone}(S_1)^\circ \cap \operatorname{cone}(S_2)^\circ$ is non-empty and $S_i \subseteq K_i$ is spanning for $i = 1, 2$, then $S_1 \cap S_2 \subseteq K_1 \cap K_2$ is a spanning embedded monoid.*

Proof. We denote by S_{12} the intersection of S_1 and S_2. Note that $S_{12} \subseteq K_1 \cap K_2$ is an embedded monoid by Lemma 4.1.13 (i). Clearly, the group generated by S_{12} is contained in $K_1 \cap K_2$. It remains to show the opposite inclusion. We denote by ι_1, ι_2 and ι_{12} the maps defined by $w \mapsto w \otimes 1$ fitting into the following diagram:

$$\begin{array}{ccccc}
K_1 & \xrightarrow{\quad\iota_1\quad} & K_1 \otimes \mathbb{Q} & \supseteq & \operatorname{cone}(S_1) \\
\cup\!\shortmid & & \cup\!\shortmid & & \\
K_1 \cap K_2 & \xrightarrow{\quad\iota_{12}\quad} & (K_1 \cap K_2) \otimes \mathbb{Q} & \supseteq & \tau := \operatorname{cone}(S_1)^\circ \cap \operatorname{cone}(S_2)^\circ \\
\cap\!\shortmid & & \cap\!\shortmid & & \\
K_2 & \xrightarrow{\quad\iota_2\quad} & K_2 \otimes \mathbb{Q} & \supseteq & \operatorname{cone}(S_2)\,.
\end{array}$$

Because of $\tau \neq \emptyset$, the rank of $K_1 \cap K_2$ and the dimension of τ coincide. Thus there are elements

$$b_1, \dots, b_r \;\in\; \iota_{12}^{-1}(\tau) \;\subseteq\; \iota_1^{-1}(\operatorname{cone}(S_1)) \cap \iota_2^{-1}(\operatorname{cone}(S_2)) \;=\; \tilde{S}_1 \cap \tilde{S}_2$$

generating $K_1 \cap K_2$ as a group. Furthermore $\tau \neq \emptyset$ implies that there is an element $x \in K_1 \cap K_2$ such that $x \otimes 1 \in \tau$ holds. Recall that $S_i \subseteq K_i$ are spanning monoids and thus Proposition 4.1.10 shows that their conductor ideals are non-empty. Since $c(\tilde{S}_i/S_i)$ contains some shifted copy of $\tilde{S}_i$, there are some $m_i \in \mathbb{Z}_{\geq 1}$, $i = 1, 2$, such that the integer multiple $m_i x$ is contained in $c(\tilde{S}_i/S_i)$, $i = 1, 2$. Hence with $m := m_1 m_2$, we have $mx \in C := c(\tilde{S}_1/S_1) \cap c(\tilde{S}_2/S_2)$. In particular, C contains the set of generators $\{mx, mx + b_1, \dots, mx + b_r\}$ for $K_1 \cap K_2$. It follows that

$$K_1 \cap K_2 \;=\; \operatorname{lin}_{\mathbb{Z}}(C) \;\subseteq\; \operatorname{lin}_{\mathbb{Z}}(c(\tilde{S}_{12}/S_{12})) \;\subseteq\; \operatorname{lin}_{\mathbb{Z}}(S_{12})$$

holds, where the inclusion in the middle was shown in Lemma 4.1.13 (iii) and the inclusion on the right-hand side follows since $c(\tilde{S}_{12}/S_{12})$ is non-empty by the same Lemma and thus contains some shifted copy of S_{12}. $\square$

Example 4.1.16. Note that without the assumption $\operatorname{cone}(S_1)^\circ \cap \operatorname{cone}(S_2)^\circ \neq \emptyset$, Proposition 4.1.15 is in general not true: For the spanning embedded monoids $\mathbb{Z}_{\geq 0}, \mathbb{Z}_{\leq 0} \subseteq \mathbb{Z}$ the intersection $\mathbb{Z}_{\geq 0} \cap \mathbb{Z}_{\leq 0} = \{0\} \subseteq \mathbb{Z}$ is not spanning.

Following ideas of Assi [**4**], we now construct an explicit point g_S of the conductor ideal of an embedded monoid $S \subseteq K$. Our setting is slightly more general than Assi's. We will make use of g_S in our investigation of Fujita's base point free conjecture for varieties with a torus action of complexity one and Picard number two in Section 4.6.

Setting 4.1.17. Consider vectors $w_1, \dots, w_r \in \mathbb{Z}^r$ being linearly independent over $\mathbb{Q}$ and let $\mathbb{Z}^r \ni w_{r+1}, \dots w_{r+t} \in \operatorname{cone}(w_1, \dots, w_r)$. For all $1 \leq j \leq t+1$, we denote by D_j the greatest common divisor of the $(r \times r)$-minors of the matrix $(w_1, \dots, w_{r+j-1})$. Let $S := \operatorname{lin}_{\mathbb{Z}_{\geq 0}}(w_1, \dots, w_{r+t})$ and $K := \operatorname{lin}_Z(w_1, \dots, w_{r+t})$.

Remark 4.1.18. In the setting of 4.1.17 the following holds:

(i) The quotient $\frac{D_j}{D_{j+1}}$ is an integer for all $1 \leq j \leq t$.

(ii) Each element $w \in K$ has a representation $w = \sum_{i=1}^{r+t} \lambda_i w_i$ with integers λ_i s.t. $0 \leq \lambda_{r+j} < D_j/D_{j+1}$ holds for all $j = 1, \dots t$.

Proof. Assertion (i) follows directly from the definition of the D_j and Assertion (ii) was proven in [**61**], Lemma 1.3. $\square$

Lemma 4.1.19. (cf. [4], Thm. 1.1) *In the setting of 4.1.17, we consider the spanning embedded monoid $S \subseteq K$ and denote by ι the map $K \to K \otimes \mathbb{Q}$, $w \mapsto w \otimes 1$. If we set*

$$g_S \;:=\; \sum_{j=1}^{t} \left(\frac{D_j}{D_{j+1}} - 1 \right) w_{r+j} - \sum_{i=1}^{r} w_i \;\in\; K\,,$$

then $g_S + \iota^{-1}(\operatorname{cone}(S)^\circ)$ is a subset of $c(\tilde{S}/S)$.

Proof. Let $v \in g_S + \iota^{-1}(\mathrm{cone}(S)^\circ)$, i.e. we have $v = g_S + u$ with some $u \in \iota^{-1}(\mathrm{cone}(S)^\circ)$ and some $v \in K$. We need to show that $v + \tilde{v}$ is contained in S, where $\tilde{v}$ denotes an arbitrary element of $\tilde{S}$. Since $v + \tilde{v}$ is an element of K, Remark 4.1.18 (ii) yields a representation

$$v + \tilde{v} \;=\; \sum_{i=1}^{r+t} \lambda_i w_i \qquad (\star)$$

with integers λ_i s.t. $0 \leq \lambda_{r+j} < \frac{D_j}{D_{j+1}}$ holds for all $j = 1, \dots t$. Together with the equality $g_S + u + \tilde{v} = v + \tilde{v}$, we obtain

$$\sum_{j=1}^{t} \left(\frac{D_j}{D_{j+1}} - 1 - \lambda_{r+j} \right) w_{r+j} + u + \tilde{v} \;=\; \sum_{i=1}^{r} (\lambda_i + 1) w_i.$$

Note that $D_j / D_{j+1} - 1 - \lambda_{r+j} \geq 0$ holds and that $\tilde{v}, w_{r+1}, \dots w_{r+t}$ define elements of $\mathrm{cone}(w_1, \dots, w_r) = \mathrm{cone}(S)$. Since u defines an element of the relative interior of $\mathrm{cone}(S)$, the same holds for the entire sum on the left-hand side. It follows that $\lambda_i + 1 > 0$ holds for all $1 \leq i \leq r$, which means that all coefficients λ_i in $(\star)$ are greater than or equal to zero. This shows that $v + \tilde{v}$ is an element of the monoid S. $\square$

4.2. Base point free monoid of non-complete toric varieties

We study the monoid of base point free Cartier divisor classes of a toric variety. In the subsequent sections, we will apply the results of this section to the toric ambient variety Z_Σ of a Mori dream space X. Since Z_Σ is in general not complete, we treat the case of non-complete toric varieties. Note that the corresponding statements for complete toric varieties and varieties arising from bunched rings are well-known; see, for instance, [**21, 3, 11, 35**].

Setting 4.2.1. Let N be a lattice and let $v_1, \dots, v_r \in N$ be pairwise different primitive vectors generating $N_{\mathbb{Q}}$ as a vector space. Set $F := \mathbb{Z}^r$ and denote by $P \colon F \to N$ the linear map sending the i-th canonical basis vector $f_i \in F$ to $v_i \in N$. Then we have mutually dual exact sequences

$$0 \longrightarrow L \longrightarrow F \xrightarrow{\;P\;} N,$$

$$0 \longleftarrow K \xleftarrow[\;Q\;]{} E \xleftarrow[\;P^*\;]{} M \longleftarrow 0\,,$$

where P^* is the dual map of P and $Q \colon E \to K := E / P^*(M)$ denotes the projection. We write $\delta \subseteq F_{\mathbb{Q}}$ and $\gamma \subseteq E_{\mathbb{Q}}$ for the respective positive orthants. Then γ is the dual cone of δ and we have the bijective face correspondence

$$\mathrm{faces}(\delta) \;\to\; \mathrm{faces}(\gamma), \qquad \delta_0 \;\mapsto\; \delta_0^* := \delta_0^\perp \cap \gamma.$$

Let Σ be a fan in N having as its one-dimensional cones the rays $\varrho_i := \mathrm{cone}(v_i)$, where $i = 1, \dots, r$. For every cone $\sigma \in \Sigma$, we denote by $\hat{\sigma} \preceq \delta$ the unique face with $P(\hat{\sigma}) = \sigma$. The *covering collection* of Σ consists of faces of $\gamma \in E_{\mathbb{Q}}$ and is given by

$$\mathrm{cov}(\Sigma) \;:=\; \{\hat{\sigma}^* \preceq \gamma;\ \sigma \in \Sigma^{\max}\}\,.$$

Now we consider the toric variety $Z = Z_\Sigma$ associated with the fan Σ. Its acting torus is $T_N := \mathrm{Spec}(\mathbb{K}[M])$. Denote by $e_1, \dots, e_r$ the canonical base vectors of E. Recall that the divisor class group of Z is given as $\mathrm{Cl}(Z) = K$, where the class of the torus-invariant prime divisor $D_i := \overline{T_N \cdot z_{\varrho_i}} \subseteq Z$ corresponding to the ray $\varrho_i \in \Sigma$ is identified with $w_i := Q(e_i) \in K$.

In the above situation, let $m \in M$. Recall that the character $\chi^m \colon T_N \to \mathbb{K}^*$ associated with m defines a rational function in $\mathbb{K}(Z)^*$. According to [**21**, Prop. 4.1.2], the principal divisor $\operatorname{div}(\chi^m)$ is given by

$$\operatorname{div}(\chi^m) \;=\; \sum_{i=1}^{r} \langle m, v_i \rangle D_i\,.$$

Furthermore, according to [**21**, Thm. 4.2.8], a Weil divisor $D = \sum a_i D_i$ is Cartier if and only if for all maximal cones $\sigma \in \Sigma$ there is $m_\sigma \in M$ such that $\langle m_\sigma, v_i \rangle = -a_i$ holds for all $v_i \in \sigma$. In this case, we have $D|_{Z_\sigma} = \operatorname{div}(\chi^{-m_\sigma})|_{Z_\sigma}$, where Z_σ denotes the affine toric variety associated with σ.

If X is a variety arising from a bunched ring $(R, \mathfrak{F}, \Phi)$ and if Σ is the fan of its minimal toric ambient variety, then we have $\operatorname{cov}(\Sigma) = \operatorname{cov}(\Phi)$.

Lemma 4.2.2. *In Setting 4.2.1, consider $w \in K$ and $\hat{\sigma}^* \in \operatorname{cov}(\Sigma)$. By Z_σ we denote the affine toric variety associated with σ. Consider a Weil divisor $D = \sum a_i D_i$ such that $w = [D]$ holds. Then the following statements are equivalent:*

(i) *We have $w \in Q(\hat{\sigma}^* \cap E)$.*

(ii) *There is $m_\sigma \in M$ such that $\operatorname{div}(\chi^{-m_\sigma})|_{Z_\sigma} = D|_{Z_\sigma}$ and $\langle m_\sigma, v_i \rangle \geq -a_i$ hold for all $1 \leq i \leq r$.*

Furthermore, if one of the statements is fulfilled, then $\operatorname{Bs}(w) \subseteq \bigcup_{v_i \notin \sigma} D_i$ holds.

Proof. Statement (i) is equivalent to the existence of an element $e_\sigma \in \hat{\sigma}^* \cap E$ such that $Q(e_\sigma) = w$ holds. The exactness of the above mutually dual sequences yields an element $m_\sigma \in M$ such that $P^*(m_\sigma) = e_\sigma - a$ holds for $a = (a_1, \ldots, a_r) \in E$. Note that

$$\begin{aligned} \langle m_\sigma, v_i \rangle &= \langle P^*(m_\sigma), f_i \rangle \\ &= \langle e_\sigma, f_i \rangle - \langle a, f_i \rangle \\ &= \langle e_\sigma, f_i \rangle - a_i \end{aligned}$$

holds. This implies that statement (i) is equivalent to the existence of an element $m_\sigma \in M$ such that $\langle m_\sigma, v_i \rangle \geq -a_i$ holds for all $1 \leq i \leq r$, with equality in case v_i is a ray of σ. Since the latter means that $\operatorname{div}(\chi^{-m_\sigma})|_{Z_\sigma}$ equals $D|_{Z_\sigma}$, we showed the equivalence of statements (i) and (ii). For the supplement note that χ^{m_σ} is a global section of the sheaf $\mathcal{O}_Z(D)$ associated with the Weil divisor D, i.e. the base locus $\operatorname{Bs}(w)$ is a subset of the support of the D-divisor

$$\operatorname{div}_D(\chi^{m_\sigma}) \;=\; \sum_{i=1}^{r} (a_i + \langle m_\sigma, v_i \rangle) D_i.$$

Furthermore, since $\langle m_\sigma, v_i \rangle = -a_i$ holds for all $1 \leq i \leq r$ with $v_i \in \sigma$, we obtain that $\operatorname{Supp}(\operatorname{div}_D(\chi^{m_\sigma})) \subseteq \bigcup_{v_i \notin \sigma} D_i$ holds. ☐

Lemma 4.2.3. *In Setting 4.2.1, let $w := [D] \in \operatorname{Cl}(Z)$ be a Weil divisor class. Then the base locus of w is given by*

$$\operatorname{Bs}(w) \;=\; \bigcap_{\substack{m \in M \\ \operatorname{div}_D(\chi^m) \geq 0}} \operatorname{Supp}(\operatorname{div}_D(\chi^m)).$$

Proof. Since $\chi^m \in \Gamma(Z, \mathcal{O}_Z(D))$ holds for all $m \in M$ with $\operatorname{div}_D(\chi^m) \geq 0$, inclusion "$\subseteq$" is obvious. For the other inclusion, let $z \in Z$ such that $z \in \operatorname{Supp}(\operatorname{div}_D(\chi^m))$ holds for all $m \in M$ with $\operatorname{div}_D(\chi^m) \geq 0$. Consider the characteristic space $p_Z \colon \widehat{Z} \to Z$ and $\hat{z} \in p_Z^{-1}(z)$ such that $H_Z \cdot \hat{z} \subseteq \hat{Z}$ is closed. Then [**3**, Corollary 1.6.2.2] shows that $\chi^m(\hat{z}) = 0$ holds for all $m \in M$ with $\operatorname{div}_D(\chi^m) \geq 0$. Since $\Gamma(X, \mathcal{O}_Z(D))$ is spanned by the characters χ^m with $\operatorname{div}_D(\chi^m) \geq 0$, this means that $f(z) = 0$ holds for all $f \in \Gamma(X, \mathcal{O}_Z(D))$. We apply again [**3**, Corollary 1.6.2.2] to see that this means $z \in \operatorname{Supp}(\operatorname{div}_D(f))$ for all $f \in \Gamma(Z, \mathcal{O}_Z(D))$, i.e. $z \in \operatorname{Bs}(w)$ holds. ☐

Definition 4.2.4. Let X be an irreducible normal prevariety. The embedded monoid of base point free Cartier divisor classes in the Picard group is called *base point free monoid of* X; we denote it by $\mathrm{BPF}(X) \subseteq \mathrm{Pic}(X)$.

The following Proposition is well-known for varieties arising from a bunched ring, see [**11, 35, 3**].

Proposition 4.2.5. *In Setting 4.2.1, we have the following statements:*

(i) *The Picard group* $\mathrm{Pic}(Z)$ *is given as a subgroup of* $\mathrm{Cl}(Z)$ *by*

$$\mathrm{Pic}(Z) \;=\; \bigcap_{\hat{\sigma}^* \in \mathrm{cov}(\Sigma)} Q(\mathrm{lin}(\hat{\sigma}^*) \cap E).$$

(ii) *The base locus of a Weil divisor class* $w \in K$ *is the following union of toric orbits* $Z(\sigma) \subseteq Z$, $\sigma \in \Sigma$:

$$\mathrm{Bs}(w) \;=\; \bigcup_{w \notin Q(\hat{\sigma}^* \cap E)} Z(\sigma).$$

(iii) *The monoid* $\mathrm{BPF}(Z)$ *of base point free Cartier divisor classes of* Z *is given by*

$$\mathrm{BPF}(Z) \;=\; \bigcap_{\hat{\sigma}^* \in \mathrm{cov}(\Sigma)} Q(\hat{\sigma}^* \cap E).$$

Proof. We prove (i). For the proof of inclusion "$\subseteq$", let $w \in \mathrm{Pic}(Z)$. This means that $w = [D]$ holds with a Cartier divisor $D = \sum a_i D_i$, i.e. for all $\sigma \in \Sigma^{\max}$ there is an element $m_\sigma \in M$ such that $\mathrm{div}(\chi^{-m_\sigma})|_{Z_\sigma} = D|_{Z_\sigma}$ holds, where Z_σ denotes the affine toric variety associated with σ. The latter is equivalent to $\langle m_\sigma, v_i \rangle = -a_i$ for all $v_i \in \sigma$. Let $e_\sigma := a - P^*(-m_\sigma)$, where we set $a := (a_1, \ldots, a_r)$. We show that $e_\sigma \in \mathrm{lin}(\hat{\sigma}^*) \cap E$ holds. Clearly, we have $e_\sigma \in E$. Furthermore, note that

$$\begin{aligned} \langle f_i, e_\sigma \rangle &= \langle f_i, a \rangle - \langle f_i, P^*(-m_\sigma) \rangle \\ &= a_i - \langle P(f_i), -m_\sigma \rangle \\ &= 0 \end{aligned}$$

holds for all $1 \le i \le r$ with $v_i \in \sigma$, i.e. e_σ is indeed an element of $\mathrm{lin}(\hat{\sigma}^*) \cap E$. We conclude that

$$w = Q(a) = Q(e_\sigma + P^*(-m_\sigma)) = Q(e_\sigma) + 0 \;\in\; Q(\mathrm{lin}(\hat{\sigma}^*) \cap E)$$

holds for all $\sigma \in \Sigma^{\max}$, where we used $Q(P^*(m_\sigma)) = 0$. For the opposite inclusion assume that w is contained in $Q(\mathrm{lin}(\hat{\sigma}^*) \cap E)$ for all maximal cones $\sigma \in \Sigma$. This means that for all $\sigma \in \Sigma^{\max}$ there is $e_\sigma \in \mathrm{lin}(\hat{\sigma}^*) \cap E$ such that $Q(e_\sigma) = w$ holds. We choose an element $a \in E$ that is contained in the fiber $Q^{-1}(w)$. Then $e_\sigma - a$ is an element of $\ker(Q) = \mathrm{im}(P^*)$, i.e. for all $\sigma \in \Sigma^{\max}$ there exists $m_\sigma \in M$ such that $P^*(m_\sigma) = e_\sigma - a$ holds. For a maximal cone $\sigma \in \Sigma^{\max}$ and the primitive generators $v_i \in \sigma$ of its rays we have the following:

$$\begin{aligned} \langle m_\sigma, v_i \rangle &= \langle P^*(m_\sigma), f_i \rangle \\ &= \langle e_\sigma, f_i \rangle - \langle a, f_i \rangle \\ &= -a_i. \end{aligned}$$

Thus the m_σ define local data for the Cartier divisor $D = \sum a_i D_i$ whose class is given by $w = [D]$, i.e. $w \in \mathrm{Pic}(Z)$ holds.

To prove assertion (ii), let $D = \sum a_i D_i$ be a Weil divisor with $[D] = w$. We first prove the inclusion "$\subseteq$". Let $z \in \mathrm{Bs}(w)$. There is exactly one cone $\sigma \in \Sigma$ with the property that $z \in Z(\sigma)$ holds. We need to show that w is not contained in $Q(\hat{\sigma}^* \cap E)$. If w was an element of $Q(\hat{\sigma}^* \cap E)$, then Lemma 4.2.2 would imply that $\mathrm{Bs}(w) \subseteq \bigcup_{v_i \notin \sigma} D_i$ holds. But since z is an element of the orbit $Z(\sigma)$, z is contained in D_i if and only if $v_i \in \sigma$ holds. This contradicts $z \in \mathrm{Bs}(w) \subseteq \bigcup_{v_i \notin \sigma} D_i$.

Thus we have $w \notin Q(\hat{\sigma}^* \cap E)$. To prove the other inclusion, let $\sigma \in \Sigma$ such that $w \notin Q(\hat{\sigma}^* \cap E)$ holds and consider an element $z \in Z(\sigma)$. We assume that z is not contained in $\mathrm{Bs}(w)$. Then there is a global section $f \in \Gamma(Z, \mathcal{O}_Z(D))$ such that $z \notin \mathrm{Supp}(\mathrm{div}_D(f))$ holds. According to Lemma 4.2.3, we may assume that $f = \chi^m$ holds with some $m \in M$ such that $\langle v_i, m\rangle \geq -a_i$ holds for all $1 \leq i \leq r$. Note that we have

$$\mathrm{div}_D(\chi^m) \;=\; \sum_{i=1}^{r}(a_i + \langle m, v_i\rangle)D_i \,.$$

Since $z \in Z(\sigma)$ is not contained in $\mathrm{Supp}(\mathrm{div}_D(\chi^m))$, we have $\langle m, v_i\rangle = -a_i$ for all $1 \leq i \leq r$ such that v_i is contained in σ. This means that $\mathrm{div}(\chi^{-m_\sigma})|_{Z_\sigma} = D|_{Z_\sigma}$ holds. Thus, Lemma 4.2.2 yields $w \in Q(\hat{\sigma}^* \cap E)$, contradicting the assumption. Hence, z is contained in $\mathrm{Bs}(w)$. Assertion (iii) is an easy consequence of (ii). □

Proposition 4.2.6. *In Setting 4.2.1, let $\sigma \in \Sigma$ be a full-dimensional maximal cone. Then the embedded monoid $Q(\hat{\sigma}^* \cap E) \subseteq Q(\mathrm{lin}(\hat{\sigma}^*) \cap E)$ is saturated. In particular, if all maximal cones of Σ are of full dimension in $N_{\mathbb{Q}}$, then $\mathrm{BPF}(Z) \subseteq \mathrm{Pic}(Z)$ is a saturated embedded monoid.*

Proof. Consider a maximal full-dimensional cone $\sigma \in \Sigma$. Then $\sigma \subseteq N_{\mathbb{Q}}$ is generated by some of the v_i's. After suitable renumbering of variables we have $\sigma = \mathrm{cone}(v_1, \ldots, v_s)$ for some $s \leq r$. The Gale dual cone $Q(\hat{\sigma}^*)$ is generated by the complementary weights w_j, that means by $w_{s+1}, \ldots, w_r$. By assumption, $v_1, \ldots, v_s$ generate $N_{\mathbb{Q}}$. According to [**10**, Lem. 8.1.(ii)], the family $(w_{s+1}, \ldots, w_r)$ thus is linearly independent in $K_{\mathbb{Q}}$. Hence we may use Lemma 4.1.6 to see that $Q(\hat{\sigma}^* \cap E) \subseteq Q(\mathrm{lin}(\hat{\sigma}^*) \cap E)$ is saturated. The supplement follows since the intersection of saturated embedded monoids is again saturated, see Lemma 4.1.13 (ii). □

Remark 4.2.7. Fujita proved in [**31**, Thm. 1], that $\mathcal{K}_X + m\mathcal{L}$ is nef for all $m \geq \dim(X) + 1$ and for all $\mathcal{L} \in \mathrm{Ample}(X) \cap \mathrm{Pic}(X)$ if X is an irreducible smooth projective variety. A result of Maeda [**51**] shows the same for irreducible normal log terminal projective varieties. In particular, irreducible normal log terminal projective varieties whose base point free monoid is saturated fulfill Fujita's base point free conjecture.

We derive the following well-known [**31, 58**] result for complete toric varieties:

Corollary 4.2.8. *If Z is a complete toric variety, then $\mathrm{BPF}(Z) \subseteq \mathrm{Pic}(Z)$ is saturated. In particular if X is log terminal and projective, then X fulfills Fujita's base point free conjecture (4.0.1).*

4.3. Base point free monoid of Mori dream spaces

In Section 4.3 we study the base point free monoid $\mathrm{BPF}(X)$ of a variety X arising from a bunched ring and show that $\mathrm{BPF}(X)$ coincides with the base point free monoid of its minimal ambient toric variety Z_Σ. This means in particular, that the study of base point free questions for varieties $X(R, \mathfrak{F}, \Phi)$ can be reduced to the study of base point free questions of non-complete toric varieties. We give criteria for $\mathrm{BPF}(X)$ to be saturated and criteria for X fulfilling Fujita's base point free conjecture, see, for instance, Corollaries 4.3.5, 4.3.8 and 4.3.16. As an application, we show in Corollaries 4.3.6, 4.3.7 and 4.3.9 that the intrinsic quadrics of the classification done in Chapter three have a saturated base point free monoid and fulfill Fujita's base point free conjecture.

Lemma 4.3.1. *Consider a variety $X = X(R, \mathfrak{F}, \Phi)$ together with its minimal toric ambient variety $Z = Z_\Sigma$ and pick a class $w \in \mathrm{Pic}(X) = \mathrm{Pic}(Z)$. Then $w \in \mathrm{Pic}(X)$ is base point free if and only if its corresponding class $w \in \mathrm{Pic}(Z)$ is base point free.*

In particular, this means that the base point free monoid of X coincides with that of Z, i.e. we have

$$\mathrm{BPF}(X) \; = \; \mathrm{BPF}(Z)\,.$$

Proof. According to Proposition 4.2.5, the base locus $B_Z(w) \subseteq Z$ of $w \in \mathrm{Pic}(Z)$ is the union of all toric orbits $Z(\sigma)$ such that $w \notin Q(\hat{\sigma}^* \cap E)$ and $\sigma \in \Sigma$ hold. Similarly, [**3**, Prop. 3.3.2.8] states that the base locus $B_X(w) \subseteq X$ of $w \in \mathrm{Pic}(X)$ is the union of all pieces $X(\hat{\sigma}^*)$ such that $w \notin Q(\hat{\sigma}^* \cap E)$ and $\hat{\sigma}^* \in \mathrm{rlv}(\Phi)$ hold. We want to show that $B_X(w)$ is non-empty if and only if $B_Z(w)$ is non-empty. Since each element $\hat{\sigma}^* \in \mathrm{rlv}(\Phi)$ corresponds to a cone $\sigma \in \Sigma$, $B_X(w) \neq \emptyset$ implies $B_Z(w) \neq \emptyset$. For the opposite direction, assume that $B_Z(w)$ is non-empty. Since the base locus of a Weil divisor is by definition a closed set, there is a cone $\sigma \in \Sigma^{\max}$ such that $w \notin Q(\hat{\sigma}^* \cap E)$ holds. But the cones $\tau \in \Sigma^{\max}$ are precisely those cones of Σ such that $\hat{\tau}^* \in \mathrm{cov}(\Phi)$ holds, which shows that $X(\hat{\sigma}^*)$ is a subset of $B_X(w)$, i.e. $B_X(w)$ is non-empty. In particular, we conclude that the base point free monoids of X and of Z coincide. □

For projective varieties, any Cartier divisor is the difference of two very ample divisors [**23**, 1.20]. Thus, the base point free monoid of projective varieties is a spanning embedded monoid. By Proposition 4.1.10, this means in particular that its conductor ideal is non-empty. For Mori dream spaces, we obtain the same result in the following Corollary. Moreover, we give a description of $\mathrm{BPF}(X)$ in terms of the covering collection and the degree map $Q\colon E \to \mathrm{Cl}(X),\ e_i \mapsto \deg(f_i)$.

Corollary 4.3.2. *The base point free monoid of $X = X(R, \mathfrak{F}, \Phi)$ is the embedded monoid given by*

$$\mathrm{BPF}(X) \; = \; \bigcap_{\gamma_0 \in \mathrm{cov}(\Phi)} Q(\gamma_0 \cap E) \; \subseteq \; \mathrm{Pic}(X)\,.$$

If X is projective, then the conductor ideal of $\mathrm{BPF}(X) \subseteq \mathrm{Pic}(X)$ *is non-empty.*

Proof. The representation of $\mathrm{BPF}(X)$ as intersection of the monoids $Q(\gamma_0 \cap E)$, $\gamma_0 \in \mathrm{cov}(\Phi)$, is an immediate consequence of Proposition 4.2.5 and Lemma 4.3.1. Note that if X is projective, then we have $\Phi = \Phi(u)$ for some ample $u \in \mathrm{Cl}(X)$. In particular, the cones $Q(\gamma_0)^\circ$, $\gamma_0 \in \mathrm{cov}(\Phi)$, intersect non-trivially. Using Proposition 4.1.15, we conclude that $\mathrm{BPF}(X) \subseteq \mathrm{Pic}(X)$ is a spanning embedded monoid. By Proposition 4.1.10, this means that its conductor ideal is non-empty. □

Example 4.3.3. Consider the surface $X = X(R, \mathfrak{F}, \Phi)$ associated with the bunched ring given by $R = \mathbb{C}[T_1, \ldots, T_4]/\langle T_1^5 T_2 + T_3^3 + T_4^2 \rangle$ with degree matrix $Q = [1, 1, 2, 3]$ and bunch defined by any divisor class $u \in \mathrm{Mov}(X)^\circ$. Denote by $e_1, \ldots, e_4$ the canonical base vectors of $E := \mathbb{Z}^4$. For indices $1 \leq \ell_1 < \ldots < \ell_s \leq 4$ we set as before

$$\gamma_{\ell_1 \ldots \ell_s} \; := \; \mathrm{cone}(e_{\ell_1}, \ldots, e_{\ell_s})\,.$$

With this, the covering collection of X is given by $\mathrm{cov}(\Phi) = \{\gamma_1, \gamma_2, \gamma_{34}\}$. Thus, the Picard group of X is $\mathbb{Z}$ and the base point free monoid of X is the monoid

$$\mathrm{BPF}(X) \; = \; Q(\gamma_1 \cap E) \cap Q(\gamma_2 \cap E) \cap Q(\gamma_{34} \cap E) \; = \; \mathrm{lin}_{\mathbb{Z}_{\geq 0}}(2, 3)\,.$$

 $\mathrm{BPF}(X) \subseteq \mathbb{Z}$

Note that the monoids $Q(\gamma_i \cap E)$, $i = 1, 2$, correspond to full-dimensional cones in the fan Σ of the minimal ambient toric variety of X and are saturated. Hence we have $\mathrm{BPF}(X) = Q(\gamma_{34} \cap E)$.

Corollary 4.3.4. *Consider a variety $X = X(R, \mathfrak{F}, \Phi)$ together with its minimal toric ambient variety $Z = Z_\Sigma$. If all maximal cones of Σ are of full dimension in $N_{\mathbb{Q}}$, then* $\mathrm{BPF}(X) \subseteq \mathrm{Pic}(X)$ *is a saturated embedded monoid.*

Corollary 4.3.5. *Consider a variety $X = X(R, \mathfrak{F}, \Phi)$ together with its minimal toric ambient variety $Z = Z_\Sigma$. If for all maximal cones of Σ that are not full-dimensional, the monoid $Q(\hat{\sigma}^* \cap E) \subseteq Q(\mathrm{lin}(\hat{\sigma}^*) \cap E)$ is saturated, then the embedded monoid $\mathrm{BPF}(X) \subseteq \mathrm{Pic}(X)$ is saturated.*

Corollary 4.3.6. *Let X be a smooth intrinsic quadric of Picard number at most two. Then $\mathrm{BPF}(X) \subseteq \mathrm{Cl}(X)$ is saturated.*

Proof. In Proposition 3.2.1 we showed that in Picard number one, there is only one smooth intrinsic quadric X per dimension with generator degrees $\deg(T_i) = 1 \in \mathrm{Cl}(X)$. In particular, $\mathrm{BPF}(X) \subseteq \mathrm{Cl}(X)$ is saturated. Now let X be a smooth intrinsic quadric of Picard number two. In Chapter three we showed that X arises from Construction 3.2.7. By going through the settings of Construction 3.2.7, we show that $\mathrm{BPF}(X) \subseteq \mathrm{Cl}(X)$ is saturated. Denote by $e_1, \ldots, e_{r+t}$ the canonical base vectors of $E := \mathbb{Z}^{r+t}$. For indices $1 \leq \ell_1 < \ldots < \ell_s \leq r + t$ we set as before

$$\gamma_{\ell_1 \ldots \ell_s} \ := \ \mathrm{cone}(e_{\ell_1}, \ldots, e_{\ell_s}).$$

In case X arises from Setting 1 or 2 in Construction 3.2.7, the covering collection of X equals

$$\{\gamma_{ij};\ 1 \leq i \leq k,\ 1 \leq j \leq t\},$$

where $k = r - 1$, r odd or $k = r$, r even hold. Note that all maximal cones of Σ are of dimension $r + t - 2$ which equals $\dim(N_\mathbb{Q})$, i.e. all maximal cones of Σ are full-dimensional. Thus $\mathrm{BPF}(X) \subseteq \mathrm{Pic}(X)$ is saturated by Corollary 4.3.4.

In case X arises from Setting 4 in Construction 3.2.7, the covering collection of X equals

$$\{\gamma_{ij};\ 1 \leq i \leq r,\ 1 \leq j \leq t,\ i \text{ odd}\} \ \cup \ \{\gamma_{ij};\ 1 \leq i, j \leq r,\ i \text{ odd}, j \text{ even},\ i + 1 \neq j\}.$$

Note that again all maximal cones of Σ are of dimension $r+t-2 = \dim(N_\mathbb{Q})$, i.e. all maximal cones of Σ are full-dimensional. Thus $\mathrm{BPF}(X) \subseteq \mathrm{Cl}(X)$ is saturated by Corollary 4.3.4.

In case X arises from Setting 3 in Construction 3.2.7, the covering collection of X contains the faces

$$\{\gamma_{ij};\ i \in \{1, 3, 4 \ldots, k\}, 1 \leq j \leq t\} \ \cup \ \{\gamma_{2i};\ 3 \leq i \leq k\},$$

where $k = r - 1$, r odd, or $k = r$, r even, hold. Note that these faces all correspond to maximal cones of Σ that are full-dimensional. If r is even, the above list of cones is exactly the covering collection of X. If r is odd, then the covering collection of X contains in addition the cone γ_{12r}, whose corresponding cone $P(\gamma_{12r}^*)$ is of dimension $r+t-3 = \dim(X)$ which is strictly smaller than $\dim(Z) = \dim(N_\mathbb{Q})$. But since $Q(\gamma_{12r} \cap E) = \mathrm{lin}_{\mathbb{Z}_{\geq 0}}((0, 1), (2, 1), (1, 1))$ holds, we conclude that $Q(\gamma_{12r} \cap E) \subseteq \mathrm{lin}_\mathbb{Z}(Q(\gamma_{12r} \cap E))$ is saturated. Corollary 4.3.5 shows that $\mathrm{BPF}(X) \subseteq \mathrm{Cl}(X)$ is saturated. ☐

Corollary 4.3.7. *Let X be a smooth intrinsic quadric of Picard number three and dimension at most four. Then $\mathrm{BPF}(X) \subseteq \mathrm{Cl}(X)$ is saturated.*

Proof. In Chapter three we showed that X is isomorphic to a variety arising from the tables in Theorems 3.3.5 and 3.3.6. According to Corollary 4.3.5, we only need to consider those members γ_0 of the covering collections that are of dimension strictly greater than $\rho(X)$. By going through the cases we conclude that $\mathrm{BPF}(X) \subseteq \mathrm{Cl}(X)$ is saturated. ☐

Corollary 4.3.8. *Consider a variety $X = X(R, \mathfrak{F}, \Phi)$ together with its minimal toric ambient variety $Z = Z_\Sigma$. If the embedded monoid $\mathrm{BPF}(X) \subseteq \mathrm{Pic}(X)$ is saturated, for instance if all maximal cones of Σ are of full dimension in $N_\mathbb{Q}$, and if one of the following criteria holds, then X fulfills Fujita's base point free conjecture, Conjecture 4.0.1.*

(i) *The variety X is projective and log terminal.*
(ii) *The divisor class $\mathcal{K}_X$ is semiample.*

Proof. Item (ii) is obvious and item (i) is a direct consequence of the result of Maeda [**51**], cf. Remark 4.2.7. □

Corollary 4.3.9. *If X is a smooth intrinsic quadric of Picard number at most two, then X fulfills Fujita's base point free conjecture, Conjecture 4.0.1, i.e. $\mathcal{K}_X + m\mathcal{L}$ is base point free for all $m \geq \dim(X) + 1$ and for all ample Weil divisor classes $\mathcal{L}$.*

We now turn to the description of $\mathrm{BPF}(X)$ in terms of the toric completions of the toric minimal ambient variety of X.

Lemma 4.3.10. *Consider a complete lattice fan (Σ, N) with minimal ray generators $v_1, \ldots, v_r$ and with the Gale dual maps $P\colon F \to N$ and $Q\colon E \to K$ as in Setting 4.2.1. For a cone $\tau \in \Sigma$, we have*

$$Q(\hat{\tau}^* \cap E) \;=\; \mathrm{lin}_{\mathbb{Z}}\big(w_i;\ e_i \in \hat{\sigma}^* \text{ for some } \sigma \in \mathrm{star}(\tau) \cap \Sigma^{\max}\big)\,.$$

Proof. Since Σ is a complete fan, the cone τ is the intersection of all cones $\sigma \in \Sigma^{\max}$ such that τ is a face of σ. This yields

$$\hat{\tau} \;=\; \bigcap_{\sigma \in \mathrm{star}(\tau) \cap \Sigma^{\max}} \hat{\sigma} \qquad .$$

Dualising implies that $\hat{\tau}^*$ is the sum of all $\hat{\sigma}^*$ such that $\sigma \in \mathrm{star}(\tau) \cap \Sigma^{\max}$ holds. Hence we observe that

$$\begin{aligned} Q(\hat{\tau}^* \cap E) \;&=\; Q\left(\Big(\sum_{\sigma \in \mathrm{star}(\tau) \cap \Sigma^{\max}} \hat{\sigma}^*\Big) \cap E\right) \\ &=\; Q\left(\mathrm{lin}_{\mathbb{Z}}(e_i;\ e_i \in \hat{\sigma}^* \text{ for some } \sigma \in \mathrm{star}(\tau) \cap \Sigma^{\max})\right) \end{aligned}$$

holds. The assertion then follows since Q is a homomorphism. □

For the remaining part of Section 4.3, we introduce the following notation:

Setting 4.3.11. Let $X = X(R, \mathfrak{F}, \Phi)$ be a projective $\mathbb{Q}$-factorial variety arising from a bunched ring $(R, \mathfrak{F}, \Phi)$. By $S := \mathrm{BPF}(X)$ we denote the base point free monoid of X, by $\tilde{S} \subseteq \mathrm{Pic}(X)$ its saturation and by $H_X = \mathrm{Spec}(\mathbb{K}[\mathrm{Cl}(X)])$ the quasitorus associated with $\mathrm{Cl}(X)$. Let $\kappa_1, \ldots, \kappa_t \in \Lambda(\overline{Z}, H_X)$ be the full-dimensional GIT-cones with $\kappa_i \subseteq \mathrm{SAmple}(X)$. By Σ_i we denote the fan arising from κ_i and by Φ_i the corresponding bunch. Note that each Σ_i contains the fan Σ of the minimal toric ambient variety Z_Σ of X as a subfan. This means that the toric varieties Z_i arising from the fans Σ_i are toric completions of Z_Σ.

Lemma 4.3.12. *In Setting 4.3.11, assume that there is $1 \leq i \leq t$ such that there is a maximal regular cone $\sigma \in \Sigma_i$. Then the embedded monoid $Q(\hat{\sigma}^* \cap E) \subseteq K = \mathrm{Cl}(X)$ is saturated and spanning.*

Proof. As a full-dimensional regular cone of the complete fan Σ_i, the number of rays of σ equals the dimension of $N_{\mathbb{Q}}$. By Gale duality, this means that $\hat{\sigma}^*$ has $\rho(X)$ rays, i.e. $Q(\hat{\sigma}^* \cap E)$ is generated by $\rho(X)$-many elements. Since κ_i is full-dimensional and sind $\kappa_i^\circ \subseteq Q(\hat{\sigma}^*)^\circ$ holds, the dimension of $Q(\hat{\sigma}^* \cap E)$ equals $\rho(X)$. Thus, Lemma 4.1.6 implies that the embedded monoid $Q(\hat{\sigma}^* \cap E) \subseteq \mathrm{lin}_{\mathbb{Z}}(Q(\hat{\sigma}^* \cap E))$ is saturated. Since σ is regular, Remark 1.3.3 tells that $Q(\hat{\sigma}^* \cap E)$ generates K as an abelian group. Hence we showed that $Q(\hat{\sigma}^* \cap E) \subseteq K$ is saturated and spanning. □

Lemma 4.3.13. *In Setting 4.3.11 consider a cone $\tau \in \Sigma^{\max}$. If there exists an index $1 \leq i \leq t$ and a regular cone $\sigma \in \Sigma_i^{\max} \cap \mathrm{star}(\tau)$, then the embedded monoid*

$Q(\hat{\tau}^* \cap E) \cap (\iota^{-1}(\kappa_i) \times K^{\mathrm{tor}}) \subseteq K$ is saturated, where ι denotes the map $K \to K_{\mathbb{Q}}$, $w \mapsto w \otimes 1$.

Proof. Lemma 4.3.12 shows that $Q(\hat{\sigma}^* \cap E) \subseteq K$ is saturated. Together with Lemma 4.1.13 (iii) we conclude that this also holds for $Q(\hat{\sigma}^* \cap E) \cap (\iota^{-1}(\kappa_i) \times K^{\mathrm{tor}})$. By Lemma 4.3.10, the weights w_i, $e_i \in \hat{\sigma}^*$, are among the generators of $Q(\hat{\tau}^* \cap E)$. This shows that $Q(\hat{\tau}^* \cap E) \cap (\iota^{-1}(\kappa_i) \times K^{\mathrm{tor}})$ is saturated. □

Corollary 4.3.14. *In Setting 4.3.11 consider a cone $\tau \in \Sigma^{\max}$. If for all $1 \le i \le t$ there exists a regular cone $\sigma_i \in \Sigma_i^{\max} \cap \mathrm{star}(\tau)$, then $Q(\hat{\tau}^* \cap E) \cap \tilde{S} \subseteq K$ is saturated.*

Proof. Lemma 4.3.13 shows that $Q(\hat{\tau}^* \cap E) \cap (\iota^{-1}(\kappa_i) \times K^{\mathrm{tor}}) \subseteq K$ is saturated for all $1 \le i \le t$. Since the saturation $\tilde{S}$ of $\mathrm{BPF}(X)$ is contained in the union of all $\kappa_i \times K^{\mathrm{tor}}$, $1 \le i \le t$, this implies that $Q(\hat{\tau}^* \cap E) \cap \tilde{S} \subseteq K$ is saturated. □

Definition 4.3.15. In Setting 4.3.11 we call X *virtually singular* if there exist $1 \le i \le t$ and $\sigma \in \Sigma^{\max}$ such that all cones in $\Sigma_i^{\max} \cap \mathrm{star}(\sigma)$ are singular.

Corollary 4.3.16. *Let X be as in Setting 4.3.11. If X is not virtually singular, then $\mathrm{BPF}(X) \subseteq K$ is saturated.*

Proof. Since X is not virtually singular, for all $1 \le i \le t$ and for all $\tau \in \Sigma^{\max}$ there is a regular cone $\sigma_{i,\tau} \in \Sigma_i^{\max} \cap \mathrm{star}(\tau)$. Corollary 4.3.14 implies that all embedded monoids $Q(\hat{\tau}^* \cap E) \cap \tilde{S}$ are saturated in K. Thus Lemma 4.1.13 (ii) completes the proof. □

Lemma 4.3.17. *Let X be as in Setting 4.3.11. If there is a relevant face $\gamma_0 \in \mathrm{rlv}(\Phi)$ such that $Q(\gamma_0 \cap E) \subseteq K$ is not saturated and the cone $Q(\gamma_0)$ has at most $\rho(X)$ rays, then each toric completion Z_i is singular.*

Proof. Since X is $\mathbb{Q}$-factorial, $Q(\gamma_0)$ is full-dimensional and has thus exactly $\rho(X)$ rays. For each ray ϱ_j of $Q(\gamma_0)$ we choose a canonical base vector e_{ϱ_j} of E such that $e_{\varrho_j} \in \gamma_0$ and $Q(e_{\varrho_j}) \in \varrho_j$ hold. By γ_1 we denote the cone generated by all vectors e_{ϱ_j}. Then γ_1 is a $\rho(X)$-dimensional face of γ_0 and $Q(\gamma_0) = Q(\gamma_1)$ holds. By assumption the embedded monoid $Q(\gamma_0 \cap E) \subseteq K$ is not saturated and thus the same holds for its submonoid $Q(\gamma_1 \cap E) \subseteq K$. Since $Q(\gamma_1 \cap E)$ is generated by $\dim(Q(\gamma_1))$-many elements, Lemma 4.1.6 implies that $Q(\gamma_1 \cap E)$ does not generate K as an abelian group, i.e. the cone $P(\gamma_1^*)$ is not regular. Note that $\mathrm{SAmple}(X) \subseteq Q(\gamma_0) = Q(\gamma_1)$ holds and each κ_i is contained in the semiample cone of X. Thus, the cone γ_1 is a relevant face for all toric completions of X arising from $\kappa_1, \ldots, \kappa_t$, which proves the statement. □

Corollary 4.3.18. *Let X be as in Setting 4.3.11. If there is $1 \le i \le t$ such that Z_i is regular, then all $Q(\gamma_0 \cap E)$ such that $\gamma_0 \in \mathrm{rlv}(\Phi)$ holds and such that $Q(\gamma_0)$ has at most $\rho(X)$ rays, are saturated.*

Corollary 4.3.19. *Let X be as in Setting 4.3.11. If X is of Picard number at most two and if there is $1 \le i \le t$ such that Z_i is regular, then $\mathrm{BPF}(X)$ is saturated in K.*

4.4. Base point free monoid of T-varieties of complexity one

The objective of this section is to give some criteria for the base point free monoid of varieties with a torus action of complexity one to be saturated, for instance in terms of big and leaf cones, see Corollary 4.4.9. As an application, we show that an irreducible smooth rational projective non-toric variety with a torus action of complexity one and of Picard number at most two has a saturated base point free monoid and in particular fulfills Fujita's base point free conjecture, see Corollary 4.4.13 and Corollary 4.4.14, respectively. In Corollary 4.4.8, we show that for $\mathbb{Q}$-factorial projective varieties with a torus action of complexity one that are not weakly tropical, the Picard group is torsion-free.

Lemma 4.4.1. *Consider a variety $X(A, P, \Phi)$ with a torus action of complexity one and consider $\gamma_0 \in \mathrm{rlv}(\Phi)$. Then the following holds for the number of generators of $Q(\gamma_0 \cap E)$ and the number of rays of $P(\gamma_0^*)$:*

	# gen. of $Q(\gamma_0 \cap E)$	*# rays of* $P(\gamma_0^*)$
$P(\gamma_0^*)$ *leaf cone,* X $\mathbb{Q}$*-factorial*	$\geq \rho(X) + r - 1$	$\leq \dim(X)$
$P(\gamma_0^*) \in \Sigma^{\max}$ *leaf cone,* X *complete and* $\mathbb{Q}$*-factorial*	$= \rho(X) + r - 1$	$= \dim(X)$
$P(\gamma_0^*)$ *big cone*	$\leq \dim(X) + \rho(X) - 2$	$\geq r + 1$
$P(\gamma_0^*)$ *elementary big cone*	$= \dim(X) + \rho(X) - 2$	$= r + 1$

Proof. For a $\mathbb{Q}$-factorial variety X, all cones of the fan of the canonical toric ambient variety are simplicial. This implies that the number of rays of a leaf cone $P(\gamma_0^*)$ is bounded from above by $\dim(X)$. Therefore the dimension of the corresponding Gale dual $\mathfrak{F}$-face γ_0 is bounded from below by $n + m - \dim(X) = r - 1 + \rho(X)$. For complete X the tropical variety $\mathrm{trop}(X)$ is contained in the support of the fan Σ of the minimal toric ambient variety of X. Since the leaves of $\mathrm{trop}(X)$ are of the same dimension as X, we conclude that the leaf cones $P(\gamma_0^*) \in \Sigma^{\max}$ have exactly $\dim(X)$-many rays if X is complete. In particular, the corresponding Gale dual $\mathfrak{F}$-faces $\gamma_0 \in \mathrm{cov}(\Phi)$ then have exactly $n + m - \dim(X) = \rho(X) + r - 1$ rays. Big cones $P(\gamma_0^*)$ have at least $r + 1$ rays, hence the number of rays of γ_0 is at most $n + m - (r + 1) = \dim(X) + \rho(X) - 2$. In case of elementary big cones, equality holds. $\square$

Lemma 4.4.2. *Consider a non-toric variety $X = X(A, P, u)$. If there exists $\gamma_0 \in \mathrm{rlv}(u)$ whose dimension is strictly greater than $\rho(X)$, then there exists a proper face $\gamma_1 \precneq \gamma_0$ such that u is contained in the relative interior of $Q(\gamma_1)$.*

Proof. We set $w_{ij} := Q(e_{ij})$, $w_k := Q(e_k)$ and

$$I \ := \ \{ij,\, k;\ e_{ij}, e_k \in \gamma_0,\ w_{ij}, w_k \text{ are contained in a ray of } \mathrm{Q}(\gamma_0)\}.$$

Now consider the cone $\sigma := \mathrm{cone}(w_{ij}, w_k;\ ij, k \in I) \subseteq K_{\mathbb{Q}}$. Since the K-grading of R is pointed, $Q(\gamma_0)$ equals σ and hence $u \in \sigma^\circ$ holds. Thus Carathéodory's Theorem implies the existence of a subset $B \subseteq I$ such that u is contained in the relative interior of $\mathrm{cone}(w_{ij}, w_k;\ ij, k, \in B)$ and such that the family $(w_{ij}, w_k;\ ij, k \in B)$ is linearly independent. The latter implies that $\gamma_1 := \mathrm{cone}(e_{ij}, e_k;\ ij, k \in B)$ is a proper face of γ_0. Hence γ_1 is as wanted. $\square$

Lemma 4.4.3. *Let $X = X(A, P, u)$ be a non-toric variety and consider $\gamma_0 \in \mathrm{cov}(u)$. If there exists a proper face $\gamma_1 \precneq \gamma_0$ such that $u \in Q(\gamma_1)^\circ$ holds, then $P(\gamma_0^*) \in \Sigma$ is a leaf cone.*

Proof. Since γ_0 is a minimal element of $\mathrm{rlv}(u)$, the face $\gamma_1 \preceq \gamma$ is not a relevant face. But $Q(\gamma_1)$ contains u in its relative interior, which means that $\gamma_1 \preceq \gamma$ is not an $\mathfrak{F}$-face. In particular, $P(\gamma_1^*)$ is not a big cone, i.e. $P(\gamma_1^*) \cap \mathrm{relint}(\lambda_{\mathrm{i}}) = \emptyset$ holds for a leaf λ_i of the tropical variety of X. Hence $P(\gamma_0^*) \preceq P(\gamma_1^*)$ implies that $P(\gamma_0^*)$ is not a big cone, either. But all $\mathfrak{F}$-faces define either big or leaf cones, so $P(\gamma_0^*)$ is a leaf cone. $\square$

Corollary 4.4.4. *Let $X = X(A, P, u)$ be a non-toric variety and consider $\gamma_0 \in \mathrm{cov}(u)$. If the dimension of γ_0 is strictly greater than $\rho(X)$, the corresponding Gale dual cone $P(\gamma_0^*)$ is a leaf cone.*

As a consequence of Corollary 4.4.4, we may enlarge the table from Lemma 4.4.1 in the case of $\mathbb{Q}$-factorial varieties $X(A, P, u)$:

Corollary 4.4.5. *Consider a non-toric $\mathbb{Q}$-factorial variety $X = X(A, P, u)$ and let $\gamma_0 \in \operatorname{cov}(u)$ such that $P(\gamma_0^*)$ is a big cone. Then the following holds:*

(i) *The dimension of γ_0 is exactly $\rho(X)$.*
(ii) *The dimension of $P(\gamma_0^*)$ is exactly $\dim(X) + r - 1$.*

In particular, $P(\gamma_0^*)$ is an elementary big cone if and only if X is a surface.

Proof. According to Corollary 4.4.4, the dimension of γ_0 is less than or equal to $\rho(X)$. Note that $\mathbb{Q}$-factoriality of X implies that $Q(\gamma_0)$ is of full dimension, hence the dimension of γ_0 is at least $\rho(X)$. Together, this proves (i). Via Gale duality the first assertion implies that the number of rays of $P(\gamma_0^*)$ is exactly $\dim(X) + r - 1$. Since X is $\mathbb{Q}$-factorial, the cone $P(\gamma_0^*)$ is simplicial, which proves the second assertion. The supplement follows since an elementary big cone has exactly $r+1$ rays. □

Corollary 4.4.6. *Let $X = X(A, P, u)$ be a non-toric $\mathbb{Q}$-factorial variety. Then the maximal big cones $\sigma \in \Sigma$ are full-dimensional. In particular, each big cone in the fan Σ of the minimal toric ambient variety of X is a face of a full-dimensional big cone $\sigma \in \Sigma^{\max}$.*

Proof. We showed in Corollary 4.4.5 that a maximal big cone is of dimension $\dim(X) + r - 1 = r + s = \dim(N_{\mathbb{Q}})$, i.e. it is full-dimensional. Since each big cone $\tau \in \Sigma$ is a face of a maximal big cone $\sigma \in \Sigma^{\max}$, the assertion follows. □

Corollary 4.4.7. *Let $X = X(A, P, u)$ be a non-toric $\mathbb{Q}$-factorial variety and consider $\gamma_0 \in \operatorname{cov}(u)$. If $Q(\gamma_0 \cap E) \cap \operatorname{Pic}(X)$ is not saturated in $\operatorname{Pic}(X)$, the corresponding Gale dual cone $P(\gamma_0^*)$ is a leaf cone.*

Proof. Since X is $\mathbb{Q}$-factorial, Corollary 4.4.6 shows that the maximal big cones of Σ are full-dimensional. Hence Proposition 4.2.6 implies that for a big cone $\sigma \in \Sigma^{\max}$, the embedded monoid $Q(\hat{\sigma}^* \cap E) \subseteq Q(\operatorname{lin}(\hat{\sigma}^*) \cap E)$ is saturated. Thus the second assertion of Lemma 4.1.13 shows that $Q(\hat{\sigma}^* \cap E) \cap \operatorname{Pic}(X) \subseteq \operatorname{Pic}(X)$ is saturated. Since the embedded monoid $Q(\gamma_0 \cap E) \cap \operatorname{Pic}(X)$ is not saturated in $\operatorname{Pic}(X)$, we conclude that $P(\gamma_0^*)$ is a leaf cone. □

Corollary 4.4.8. *Let $X = X(A, P, u)$ be a non-toric variety being not weakly tropical. Then the following hold:*

(i) *If X is $\mathbb{Q}$-factorial, the Picard group $\operatorname{Pic}(X)$ of X is torsion-free.*
(ii) *If X is locally factorial, the class group $\operatorname{Cl}(X)$ of X is torsion-free.*

Proof. Since X is not weakly tropical, there exists a big cone τ in the fan Σ of the minimal toric ambient variety Z of X. If X is $\mathbb{Q}$-factorial, Corollary 4.4.6 shows that τ is contained in a full-dimensional big cone $\sigma \in \Sigma$. The existence of a full-dimensional cone in the fan Σ implies that the Picard group of Z is torsion-free. Since the Picard groups of X and Z coincide, this yields the first assertion. To prove the second item, note that via Gale duality, σ corresponds to a relevant face $\gamma_0 \in \operatorname{rlv}(u)$ having dimension $\rho(X)$. This means that the monoid $Q(\gamma_0 \cap E) \subseteq \operatorname{Cl}(X)$ has $\rho(X)$ generators. If X is locally factorial, then $Q(\gamma_0 \cap E)$ generates $\operatorname{Cl}(X)$ as an abelian group, i.e. $\operatorname{Cl}(X)$ is generated by $\rho(X)$-many elements. We conclude that $\operatorname{Cl}(X)$ is torsion-free. □

Corollary 4.4.9. *Let $X = X(A, P, u)$ be a non-toric $\mathbb{Q}$-factorial variety and denote by Σ the fan of the minimal toric ambient variety of X. If one of the following equivalent conditions is fulfilled, then $\operatorname{BPF}(X) \subseteq \operatorname{Pic}(X)$ is saturated:*

(i) *The set $\Sigma^{\max}$ contains no leaf cones.*
(ii) *Each leaf cone $\tau \in \Sigma$ is a face of a big cone $\sigma \in \Sigma$.*
(iii) *The covering collection $\operatorname{cov}(u)$ consists of $\rho(X)$-dimensional cones.*
(iv) *$\Sigma^{\max}$ consists of cones having $\dim(X) + r - 1$ rays.*

Proof. The fact that $\mathrm{BPF}(X) \subseteq \mathrm{Pic}(X)$ is saturated if condition (i) holds is an immediate consequence of Corollary 4.4.7. The equivalence of (i) and (ii) is obvious. Via Gale duality, the members of the covering collection correspond to the cones of $\Sigma^{\max}$, which proves the equivalence of (iii) and (iv). The implication "(i)⇒(iv)" was proven in Corollary 4.4.5, while the reverse was shown in Lemma 4.4.1. □

Corollary 4.4.10. *Let $X = X(A, P, u)$ be a non-toric $\mathbb{Q}$-factorial variety and denote by Σ the fan of the minimal toric ambient variety of X. If one of the criteria of Corollary 4.4.9 together with one of the following criteria is fulfilled, then X fulfills Fujita's base point free conjecture, Conjecture 4.0.1.*

(i) *The variety X is projective and log terminal.*
(ii) *The canonical class $\mathcal{K}_X$ is semiample.*

Corollary 4.4.11. *Let $X = X(A, P, u)$ be a non-toric $\mathbb{Q}$-factorial variety and denote by Σ the fan of the minimal toric ambient variety. If for all $0 \le i \le r$*

$$\sum_{\substack{0\le\ell\le r\\ \ell\neq i}} n_\ell \ \ge\ \rho(X) + r$$

holds, then Σ contains no maximal leaf cone. In particular, $\mathrm{BPF}(X) \subseteq \mathrm{Pic}(X)$ is then saturated.

Proof. Assume that there was a leaf cone $\sigma \in \Sigma^{\max}$. Lemma 4.4.1 explains that the face $\gamma_0 \in \mathrm{cov}(u)$ with $\sigma := P(\gamma_0^*)$ has exactly $\rho(X) + r - 1$ rays. Note that σ is contained in a leaf λ_{i_0} of $\mathrm{trop}(X)$. Hence $\mathrm{cone}(e_{ij}) \preceq \gamma_0$ holds for all $0 \le i \le r$, $i \ne i_0$, $1 \le j \le n_i$. This means that

$$n_0 + \ldots + n_{i_0-1} + n_{i_0+1} + \ldots + n_r \ \le\ \rho(X) + r - 1$$

holds, contradicting the formula in the Corollary. We conclude that Σ contains no maximal leaf cone. Thus Corollary 4.4.9 completes the proof. □

Corollary 4.4.12. *Let $X = X(A, P, u)$ be a non-toric $\mathbb{Q}$-factorial variety and denote by Σ the fan of the minimal toric ambient variety. If at least $\rho(X) + 1$ monomials $T_i^{l_i}$ of the relations $g_0, \ldots, g_{r-2}$ contain strictly more than one variable, then Σ contains no maximal leaf cone. In particular, $\mathrm{BPF}(X) \subseteq \mathrm{Pic}(X)$ then is saturated.*

Proof. Choose an index $0 \le i \le r$. By assumption, at least $\rho(X)$ elements of the set $\{n_0, \ldots, n_{i-1}, n_{i+1}, \ldots, n_r\}$ are strictly greater than one. Furthermore, the remaining $r - \rho(X)$ elements are greater than or equal to one. Hence the sum on the left-hand side of Corollary 4.4.11 is at least $2\rho(X) + (r - \rho(X))$, which equals $\rho(X) + r$. Thus Corollary 4.4.11 completes the proof. □

Corollary 4.4.13. *If X is an irreducible smooth rational projective non-toric variety of Picard number at most two admitting a torus action of complexity one, then the base point free monoid of X is saturated.*

Proof. In Picard number one, by a result of Liendo and Süß [**49**], X is either a three- or a four-dimensional full intrinsic quadric with generator degrees $\deg(T_i) = 1 \in \mathrm{Cl}(X)$, i.e. the base point free monoid of X is saturated.

In Picard number two, all irreducible smooth rational projective non-toric varieties are isomorphic to a variety listed in Theorem 2.1.1. Note that according to Corollary 4.4.12, varieties Nos. 1, 2, 4, 5, 6, 7, 8, 9 and 13 have a saturated base point free monoid. Corollary 4.4.7 shows that for Nos. 3, 10, 11 and 12 it is sufficient to consider the leaf cones $P(\gamma_0^*)$, $\gamma_0 \in \mathrm{cov}(u)$. For varieties Nos. 3, 11 and 12, there are no leaf cones in $\Sigma^{\max}$. For variety No. 10, the only leaf cone in $\Sigma^{\max}$ is $P(\gamma_{125}^*)$, where we set as before

$$\gamma_{\ell_1\ldots\ell_s} \ :=\ \mathrm{cone}(e_{\ell_1}, \ldots, e_{\ell_s})$$

with the canonical base vectors $e_i \in E = \mathbb{Z}^{n+m}$. Since $Q(\gamma_{125} \cap E)$ defines a saturated monoid in the class group of variety No. 10, the base point free monoid of this variety is saturated. □

Corollary 4.4.14. *If X is an irreducible smooth rational projective non-toric variety of Picard number at most two admitting a torus action of complexity one, then X fulfills Fujita's base point free conjecture, Conjecture 4.0.1.*

4.5. Base point free questions for T-varieties of complexity one and Picard number one

We investigate the base point free monoid and Fujita's base point free conjecture, Conjecture 4.0.1, for irreducible rational non-toric T-varieties of complexity one and Picard number one. If X is smooth, then by a result of Liendo and Süß [**49**], X is either a three- or a four-dimensional full intrinsic quadric with generator degrees $\deg(T_i) = 1 \in \mathrm{Cl}(X)$. Thus its base point free monoid is saturated and X fulfills Fujita's base point free conjecture, Conjecture 4.0.1. In this section, we generalize this result to the singular case: In Theorem 4.5.5, we use Frobenius numbers to show that rational non-toric Gorenstein varieties $X(A, P, u)$ with $\mathrm{Cl}(X) = \mathbb{Z}$ fulfill Fujita's base point free conjecture.

Example 4.5.1. Here we give an example of a series of locally factorial non-toric rational varieties X with a torus action of complexity one and non-saturated base point free monoid $\mathrm{BPF}(X) \subseteq \mathrm{Pic}(X) = \mathbb{Z}$. Let $x_1, x_2 \in \mathbb{Z}_{\geq 2}$ be coprime integers, i.e. there exist integers $a_1, a_2 \in \mathbb{Z}$ with $-1 = a_1 \cdot x_1 + a_2 \cdot x_2$, and set $y := x_1 \cdot x_2 - 1$. Consider the matrices

$$P = \begin{bmatrix} -y & -1 & x_2 & 0 \\ -y & -1 & 0 & x_1 \\ 1 & 0 & a_1 & a_2 \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix},$$

the graded ring $R := R(A, P)$ and the surface $X := X(A, P, u)$ defined by any element $u \in \mathrm{Mov}(R)^\circ$. The grading of R and the covering collection of X are given by

$$Q = \begin{bmatrix} 1 & 1 & x_1 & x_2 \end{bmatrix} \quad \text{and} \quad \mathrm{cov}(u) = \{\gamma_{34}, \gamma_1, \gamma_2\},$$

where we set as before $\gamma_{\ell_1 \ldots \ell_s} := \mathrm{cone}(e_{\ell_1}, \ldots, e_{\ell_s})$ for the canonical base vectors $e_i \in \mathbb{Z}^4$. The base point free monoid of X is the numerical monoid $\mathrm{BPF}(X) = \mathrm{lin}_{\mathbb{Z}_{\geq 0}}(x_1, x_2)$ and we can use Sylvester's formula, Proposition 4.1.1, to compute its Frobenius number:

$$\mathcal{F}(\mathrm{BPF}(X)) = x_1 \cdot x_2 - x_1 - x_2 .$$

In particular, for any arbitrary natural number $m \in \mathbb{Z}_{\geq 0}$ there exists a $\mathbb{C}^*$-surface X whose global bound $n_X \in \mathbb{Z}_{\geq 0}$ such that nw is base point free for all ample divisor classes $w \in \mathrm{Cl}(X)$ and all $n \geq n_X$ is bigger than m.

Remark 4.5.2. Let X be any irreducible normal quasi-projective variety. If X fulfills Fujita's base point free conjecture, Conjecture 4.0.1, then X is Gorenstein. This means that in Proposition 4.5.4 and Theorem 4.5.5, it is no additional restriction to assume that X is Gorenstein.

Proof. Consider $m \geq \dim(X) + 1$ and $\mathcal{L} \in \mathrm{Ample}(X) \cap \mathrm{Pic}(X)$. If X fulfills Fujita's base point free conjecture, then $\mathcal{K}_X + m\mathcal{L}$ is base point free. In particular, we obtain $\mathcal{K}_X + m\mathcal{L} = \mathcal{L}'$ for some $\mathcal{L}' \in \mathrm{Pic}(X)$. Thus, $\mathcal{K}_X = \mathcal{L}' - m\mathcal{L}$ is contained in the Picard group of X. □

Remark 4.5.3. Let X be a variety arising from a bunched ring. Denote by $\mathcal{L}_1, \ldots, \mathcal{L}_s$ the ample elements of a Hilbert basis of the monoid of semiample Cartier divisor classes. Note that as a consequence of Corollary 4.3.2, a sufficient criterion for X fulfilling Fujita's base point conjecture, Conjecture 4.0.1, is that $\mathcal{K}_X + (\dim(X)+1)\mathcal{L}_i$ is an element of the conductor ideal of the base point free monoid $\mathrm{BPF}(X) \subseteq \mathrm{Pic}(X)$ for all $\mathcal{L}_1, \ldots, \mathcal{L}_s$.

Proposition 4.5.4. *Consider a non-toric Gorenstein variety $X = X(A, P, u)$. If $\rho(X) = 1$ holds and there are at least two monomials $T_i^{l_i}$ with $n_i \geq 2$, then $\mathcal{K}_X + m\mathcal{L}$ is base point free for all ample Cartier divisor classes $\mathcal{L}$ and for all $m \geq \dim(X)+1$, i.e. X fulfills Fujita's base point free conjecture, Conjecture 4.0.1.*

Proof. After suitable admissible operations, there is $1 \leq x \leq r$, $x \geq 2$, such that $n_0, \ldots, n_x \geq 2$ and $n_{x+1}, \ldots, n_r = 1$ hold. We may apply Corollary 4.4.12 to see that the covering collection of X consists of big cones. To be precise, we have

$$\mathrm{cov}(u) \;=\; \{\mathrm{cone}(e_{ij}),\, \mathrm{cone}(e_k)\,;\; 0 \leq i \leq x,\; 1 \leq j \leq n_i,\; 1 \leq k \leq m\}\,.$$

This implies that the Picard group of X is given by

$$\mathrm{Pic}(X) \;=\; \bigcap_{\substack{0\leq i\leq x\\ 1\leq j\leq n_i}} \mathrm{lin}_{\mathbb{Z}}(w_{ij}) \;\cap\; \bigcap_{1\leq k\leq m} \mathrm{lin}_{\mathbb{Z}}(w_k)\,.$$

Since the grading of $R(A,P)$ is pointed, we may assume that $w_{ij}^0, w_k^0 \in \mathbb{Z}_{>0}$ hold. Note that $\mathrm{lin}_{\mathbb{Z}}(w_{ij})$, $0 \leq i \leq x$, and $\mathrm{lin}_{\mathbb{Z}}(w_k)$, $1 \leq k \leq m$, are free $\mathbb{Z}$-modules of rank one isomorphic to $w_{ij}^0\mathbb{Z}$ and to $w_k^0\mathbb{Z}$, respectively. Since $\mathbb{Z}$ is a principal domain and $\mathrm{Pic}(X)$ is a submodule of the finitely generated free module $\mathrm{lin}_{\mathbb{Z}}(w_{01})$ of rank one, we conclude that $\mathrm{Pic}(X)$ is a free $\mathbb{Z}$-module. This means that $\mathrm{Pic}(X) = \mathrm{lin}_{\mathbb{Z}}(L)$ holds with some $L \in \mathrm{Pic}(X)$, $L^0 \in \mathbb{Z}_{>0}$. In order to show that X fulfills Fujita's base point free conjecture, it is thus enough to show that $P := \mathcal{K}_X + (\dim(X)+1)\, L$ is contained in the conductor ideal of the embedded monoid $\mathrm{BPF}(X) \subseteq \mathrm{Pic}(X)$. Note that by the above formula for $\mathrm{Pic}(X)$, w_{ij}^0, $0 \leq i \leq x$, and w_k^0, $1 \leq k \leq m$, divide L^0 and thus $w_{ij}^0, w_k^0 \leq L^0$ holds for $0 \leq i \leq x$, $1 \leq k \leq m$ $(\star)$. Furthermore Corollary 4.4.9 shows that the embedded monoid $\mathrm{BPF}(X) \subseteq \mathrm{Pic}(X)$ is saturated. In order to prove that X fulfills Fujita's base point free conjecture, it hence remains to show that P^0 is strictly greater than zero. Note that $w_{i1} \leq \deg(g_0)/2$ holds for $x+1 \leq i \leq r$ since X is non-toric. Furthermore we have $\mathcal{K}_X = (r-1)\deg(g_0) - \sum w_{ij} - \sum w_k$. Together with $\dim(X)+1 = \sum_{i=0}^{x} n_i - x + m + 1$ and $(\star)$, we obtain

$$\begin{aligned}
P^0 \;&=\; (r-1)\deg(g_0)^0 \;-\; \sum_{\substack{0\leq i\leq x\\ 1\leq j\leq n_i}} w_{ij}^0 \;-\; \sum_{i=x+1}^{r} w_{j1}^0 \;-\; \sum_{k=1}^{m} w_k^0 \\
&\qquad +\; (\sum_{i=0}^{x} n_i - x + m + 1)\, L^0 \\
&\geq\; (\frac{r}{2} + \frac{x}{2} - 1)\deg(g_0)^0 \;-\; \sum_{\substack{0\leq i\leq x\\ 1\leq j\leq n_i}} w_{ij}^0 \;+\; (\sum_{i=0}^{x} n_i - x + 1)\, L^0\,.
\end{aligned}$$

We distinguish the following two cases:

(i) There is an index $0 \leq i \leq x$, $1 \leq j \leq n_0$, with $w_{ij}^0 = L^0$.
(ii) We have $w_{ij}^0 < L^0$ for all $0 \leq i \leq x$, $1 \leq j \leq n_0$.

In the first case, i.e. if there is an index $0 \le i \le x$, $1 \le j \le n_0$, with $w_{ij}^0 = L^0$, the free part $\sum_{j=1}^{n_i} l_{ij} w_{ij}^0$ of $\deg(g_0)$ is strictly greater than L^0. We obtain

$$\begin{aligned} P^0 &\ge (\frac{r}{2} + \frac{x}{2} - 1)(L^0 + 1) - \sum_{\substack{0 \le i \le x \\ 1 \le j \le n_i}} w_{ij}^0 + (\sum_{i=0}^{x} n_i - x + 1) L^0 \\ &\ge (\frac{r}{2} + \frac{x}{2} - 1)(L^0 + 1) + (-x + 1) L^0 \\ &= (\frac{r}{2} + \frac{x}{2} - 1 - x + 1) L^0 + (\frac{r}{2} + \frac{x}{2} - 1) \\ &> 0, \end{aligned}$$

where the last inequality is true since r is strictly greater than x. Now we treat the second case, i.e. we assume that $w_{ij}^0 < L^0$ holds for all $0 \le i \le x$, $1 \le j \le n_0$. According to $(\star)$ we obtain $w_{ij}^0 \le L^0/2$ for all $0 \le i \le x$. With this and with $n_0, \dots, n_x \ge 2$ we then obtain

$$\begin{aligned} P^0 &\ge (\frac{r}{2} + \frac{x}{2} - 1) \deg(g_0)^0 + (n_0 + \ldots + n_x) \frac{L^0}{2} + (-x + 1) L^0 \\ &\ge 0 + (2(x+1)) \frac{L^0}{2} + (-x + 1) L^0 \\ &= 2L^0 . \end{aligned}$$

Since $L^0 > 0$ holds, this completes the proof. □

Theorem 4.5.5. *Let $X = X(A, P, u)$ be a non-toric variety. If $\mathrm{Cl}(X) = \mathbb{Z}$ holds and if X is Gorenstein, then $\mathcal{K}_X + m\mathcal{L}$ is base point free for all ample Cartier divisor classes $\mathcal{L}$ and for all $m \ge \dim(X) + 1$, i.e. X fulfills Fujita's base point free conjecture, Conjecture 4.0.1.*

Proof. After suitable admissible operations, we have $n_0, \dots, n_x \ge 2$ as well as $n_{x+1}, \dots, n_r = 1$ for some $0 \le x \le r$. Note that since the grading of $R(A, P)$ is pointed, we may assume that $w_{ij}, w_k > 0$ hold. Furthermore, since $\mathrm{Cl}(X) = \mathbb{Z}$ holds, [**3**, Theorem 4.2.3 (iv)] implies that the exponents l_{i1}, where $x + 1 \le i \le r$, are pairwise coprime. Since all monomials $T_i^{l_i}$ are $\mathrm{Cl}(X)$-homogeneous of the same degree, we conclude that

$$w_{l1} = \alpha \prod_{\substack{i=x+1 \\ i \ne l}}^{r} l_{i1}$$

holds for all $x + 1 \le l \le r$ with some $\alpha \in \mathbb{Z}$. In particular, the degree of the relations g_i is given as $\deg(g_0) = \alpha\, l_{x+1,1} \cdots l_{r1}$. Let L be the Picard index of X, i.e. we set $L := [\mathrm{Cl}(X) : \mathrm{Pic}(X)]$. In order to prove that X fulfills Fujita's base point free conjecture, it is sufficient to show that $P := \mathcal{K}_X + (\dim(X) + 1)L$ is contained in the conductor ideal of $\mathrm{BPF}(X) \subseteq \mathrm{Pic}(X)$. We will show that this is true in each of the three cases $x = -1$, $x = 0$ and $x \ge 1$.

If $x = -1$ holds, then we have $n_0 = \ldots = n_r = 1$ and the covering collection of X is given by

$$\mathrm{cov}(u) = \{\mathrm{cone}(e_k),\ \tau_\ell;\ 1 \le k \le m,\ 0 \le \ell \le r\},$$

where $\tau_\ell := \mathrm{cone}(e_{i1};\ 0 \le i \le r,\ i \ne \ell)$ holds. This implies that the Picard index of X is $L = \mathrm{lcm}(w_k,\ \alpha \prod_{i=0}^{r} l_{i1};\ 1 \le k \le m) \in \mathbb{Z}_{>0}$. Note that X is $\mathbb{Q}$-factorial. Thus Corollary 4.4.7 shows that the embedded monoids

$$Q(\mathrm{cone}(e_k) \cap E) \ \cap\ \mathrm{Pic}(X) \ \subseteq\ \mathrm{Pic}(X)$$

are saturated. In order to show that P is contained in the conductor ideal of the embedded monoid $\mathrm{BPF}(X) \subseteq \mathrm{Pic}(X)$, it is according to Lemma 4.1.13 (iii) sufficient to show that P is contained in the conductor ideals of $Q(\tau_\ell \cap E) \subseteq Q(\mathrm{lin}(\tau_\ell) \cap E)$, where $0 \leq \ell \leq r$ holds. Note that the largest element of $Q(\mathrm{lin}(\tau_\ell) \cap E)$ that is not contained in the conductor ideal of $Q(\tau_\ell \cap E)$ is

$$c_\ell \;:=\; \alpha\, l_{\ell 1}\, \mathcal{F}\left(\prod_{\substack{i=0 \\ i \neq j,\ell}}^{r} l_{i1}\ ;\ \ 0 \leq j \leq r,\ j \neq \ell \right).$$

We apply Lemma 4.1.4 to see that

$$\begin{aligned} c_\ell \;&=\; \alpha\, l_{\ell 1} \left((r-1) \prod_{\substack{i=0 \\ i \neq \ell}}^{r} l_{i1} \;-\; \sum_{\substack{i=0 \\ i \neq \ell}}^{r} \prod_{\substack{j=0 \\ j \neq \ell, i}}^{r} l_{j1} \right) \\ &=\; (r-1)\deg(g_0) \;-\; \sum_{\substack{i=0 \\ i \neq \ell}}^{r} \alpha \prod_{\substack{j=0 \\ j \neq i}}^{r} l_{j1} \end{aligned}$$

holds. Note that we have $\dim(X)+1 = m+2$. Since w_k divides L and thus $w_k \leq L$ holds, we obtain

$$\begin{aligned} P \;&=\; (r-1)\deg(g_0) \;-\; \sum_{i=0}^{r} \alpha \prod_{\substack{j=0 \\ j \neq i}}^{r} l_{j1} \;-\; \sum_{k=1}^{m} w_k \;+\; (m+2)L \\ &\geq\; (r-1)\deg(g_0) \;-\; \sum_{i=0}^{r} \alpha \prod_{\substack{j=0 \\ j \neq i}}^{r} l_{j1} \;+\; 2L \\ &=\; c_\ell \;-\; \alpha \prod_{\substack{j=0 \\ j \neq \ell}}^{r} l_{j1} \;+\; 2L\,. \end{aligned}$$

Recall that X is non-toric and thus the exponents l_{i1} are strictly greater than one. Furthermore, we obtain

$$\alpha \prod_{\substack{j=0 \\ j \neq \ell}}^{r} l_{j1} \;\leq\; \frac{1}{2} L\,,$$

which proves $P \geq c_\ell + 3/2L > c_\ell$. As argued above, this shows that P is contained in the conductor ideal of $\mathrm{BPF}(X) \subseteq \mathrm{Pic}(X)$ if $x = -1$ holds.

If $x = 0$ holds, then the covering collection of X is given by

$$\mathrm{cov}(u) \;=\; \{\mathrm{cone}(e_{0j}),\, \mathrm{cone}(e_k),\, \mathrm{cone}(e_{11}, \ldots, e_{r1})\,;\ \ 1 \leq k \leq m,\ 1 \leq j \leq n_0\}\,.$$

In particular, w_{0j}, $1 \leq j \leq n_0$, and w_k, $1 \leq k \leq m$, divide L. Note that we have $L = \mathrm{lcm}(w_{0j}, w_k;\ 1 \leq k \leq m,\ 1 \leq j \leq n_0) \in \mathbb{Z}_{>0}$. Since X is $\mathbb{Q}$-factorial, Corollary 4.4.7 and Lemma 4.1.13 (iii) show that the embedded monoids

$$Q(\mathrm{cone}(e_{0j}) \cap E) \;\cap\; \mathrm{Pic}(X), \;\; Q(\mathrm{cone}(e_k) \cap E) \;\cap\; \mathrm{Pic}(X) \;\subseteq\; \mathrm{Pic}(X)$$

are saturated. This means that in order to show that P is contained in the conductor ideal of the embedded monoid $\mathrm{BPF}(X) \subseteq \mathrm{Pic}(X)$, Lemma 4.1.13 (iii) shows that is sufficient to prove that P is contained in the conductor ideal of the embedded monoid

$$Q(\mathrm{cone}(e_{11}, \ldots, e_{r1}) \cap E) \;\subseteq\; Q(\mathrm{lin}(\mathrm{cone}(e_{11}, \ldots, e_{r1})) \cap E)\,.$$

The largest element of $Q(\mathrm{lin}(\mathrm{cone}(e_{11}, \ldots, e_{r1})) \cap E)$ that is not contained in the conductor ideal of this monoid is

$$c \;:=\; \alpha\, \mathcal{F}\left(\prod_{\substack{i=1 \\ i \neq j}}^{r} l_{i1}\ ;\ \ 1 \leq j \leq r \right) \;=\; (r-1)\deg(g_0) \;-\; \sum_{i=1}^{r} \alpha \prod_{\substack{j=1 \\ j \neq i}}^{r} l_{j1}\,,$$

where the second equality holds according to Lemma 4.1.4. Note that we have $\dim(X) + 1 = n_0 + m + 1$. Since w_{0j} and w_k divide L and thus $w_{0j}, w_k \leq L$ holds, we obtain the following:

$$
\begin{aligned}
P &= (r-1)\deg(g_0) - \sum_{j=1}^{r} w_{0j} - \sum_{i=1}^{r} \alpha \prod_{\substack{j=1\\ j\neq i}}^{r} l_{j1} - \sum_{k=1}^{m} w_k + (n_0+m+1)\,L \\
&\geq (r-1)\deg(g_0) - \sum_{i=1}^{r} \alpha \prod_{\substack{j=1\\ j\neq i}}^{r} l_{j1} + L \\
&= c + L\,.
\end{aligned}
$$

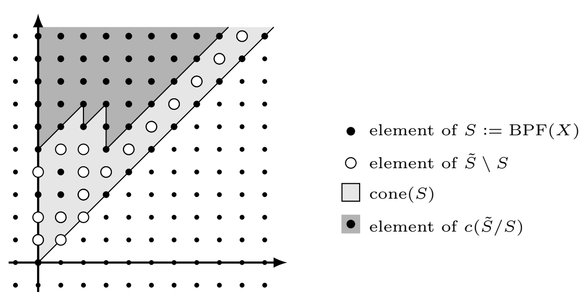
Because of $L > 0$, the above computation shows that $P > c$ holds. This proves that $P = \mathcal{K}_X + (\dim(X)+1)L$ is contained in the conductor ideal of $\mathrm{BPF}(X) \subseteq \mathrm{Pic}(X)$ if $x = 0$ holds.

Now we treat the final case $x \geq 1$. Here we may apply Proposition 4.5.4 to see that X fulfills Fujita's base point free conjecture. □

Remark 4.5.6. The statement of Theorem 4.5.5 is not true for higher Picard numbers, see, for instance, Example 4.9.6.

4.6. Base point free questions for T-varieties of complexity one and Picard number two

We investigate the base point free monoid and Fujita's base point free conjecture, Conjecture 4.0.1, for varieties with a torus action of complexity one and Picard number two. Although there are in general semiample divisor classes that are not base point free, Proposition 4.6.3 shows that a non-toric locally factorial variety $X = X(A, P, u)$ that is of Picard number two fulfills Fujita's base point free conjecture (4.0.1) if and only if the same statement holds with base point free replaced by semiample, i.e. if $\mathcal{K}_X + m\mathcal{L}$ is semiample for all $m \geq \dim(X) + 1$ and for all ample Weil divisor classes $\mathcal{L}$. Hence in this case Fujita's base point free conjecture is a question of convex geometry rather than of monoid membership.



Example 4.6.1. Here we give an example of a locally factorial projective variety whose base point free monoid $\mathrm{BPF}(X)$ is not saturated. Consider the matrices

$$
P = \begin{bmatrix} -7 & -2 & 3 & 0 & 0 \\ -7 & -2 & 0 & 10 & 1 \\ -3 & -1 & 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -1 & 0 \end{bmatrix}
$$

as well as the graded ring $R := R(A, P)$ and the variety $X := X(A, P, u)$ defined by the Weil divisor class $(1, 2) \in \mathrm{Mov}(R)^{\circ}$. The grading of R and the covering collection of X are given by

$$Q = \begin{bmatrix} 1 & -2 & 1 & 0 & 3 \\ 1 & 1 & 3 & 1 & -1 \end{bmatrix} \quad \text{and} \quad \mathrm{cov}(u) = \{\gamma_{123}, \gamma_{345}, \gamma_{25}, \gamma_{14}\}.$$

This yields $\mathrm{SAmple}(X) = \mathrm{cone}((1,1),(0,1))$ and the base point free monoid is the intersection of $Q(\gamma_{123} \cap E)$ and $Q(\gamma_{345} \cap E)$ illustrated in the above picture. Note that X is locally factorial, i.e. $\mathrm{Pic}(X) = \mathrm{Cl}(X) = \mathbb{Z}^2$ holds. As the above figure indicates, the embedded monoid $\mathrm{BPF}(X) \subseteq \mathrm{Pic}(X)$ is not saturated; for instance $(0,1) \in \mathrm{Cl}(X)$ is semiample but not base point free. The region shaded in dark gray indicates the conductor ideal of $\mathrm{BPF}(X) \subseteq \mathrm{Pic}(X)$. According to Proposition 4.6.3, the variety X fulfills Fujita's base point free conjecture. This is indeed the case since $\mathcal{K}_X + (\dim(X)+1)(1,2) = (0,4) + 3\,(1,2)$ is contained in the conductor ideal of the base point free monoid.

In the following we consider non-toric locally factorial varieties $X = X(A,P,u)$ of complexity one and of Picard number two. Recall that according to Remark 1.3.3, local factoriality of X implies that for all relevant faces $\gamma_0 \in \mathrm{rlv}(u)$, the embedded monoid $Q(\gamma_0 \cap E) \subseteq \mathrm{Cl}(X)$ is spanning. Note that Lemma 2.4.3 implies that the fan Σ contains a big cone. Hence Corollary 4.4.8 (ii) shows that $\mathrm{Cl}(X)$ is torsion-free, i.e. $\mathrm{Cl}(X) = \mathbb{Z}^2$ holds. We will frequently work with the canonical base vectors $e_{ij}, e_k \in E = \mathbb{Z}^{n+m}$ and the faces

$$\gamma_{i_1j_1,\ldots,i_aj_a,k_1,\ldots,k_b} \;:=\; \mathrm{cone}(e_{i_1j_1},\ldots,e_{i_aj_a},e_{k_1},\ldots,e_{k_b}) \;\preceq\; \gamma$$

of the positive orthant $\gamma = \mathbb{Q}^{n+m}_{\geq 0}$. With $w_{ij} = Q(e_{ij})$ and $w_k = Q(e_k)$, the columns of the $2 \times (n+m)$ degree matrix Q will be written as

$$w_{ij} \;=\; (w_{ij}^1, w_{ij}^2) \;\in\; \mathbb{Z}^2 \qquad \text{and} \qquad w_k \;=\; (w_k^1, w_k^2) \;\in\; \mathbb{Z}^2.$$

Lemma 4.6.2. *Let $X = X(A,P,u)$ be a non-toric locally factorial variety of Picard number two such that $n_{x+1} = n_{x+2} = \ldots = n_r = 1$ holds for some $0 \leq x < r$. Then the following hold:*

(i) *There are $\alpha, \beta \in \mathbb{Z}$ such that for all $i = x+1,\ldots,r$, the l_{i1} are pairwise coprime and we have*

$$w_{i1}^1 \;=\; \alpha \prod_{\substack{\ell=x+1 \\ \ell\neq i}}^{r} l_{\ell 1}, \qquad w_{i1}^2 \;=\; \beta \prod_{\substack{\ell=x+1 \\ \ell\neq i}}^{r} l_{\ell 1}.$$

(ii) *If* $\mathrm{cone}(e_{x1}, e_{x2}, e_{i1};\ i = x+1,\ldots,r)$ *is a relevant face and $n_x = 2$ as well as $w_{x1} = (1,0)$ hold, then $\gcd(l_{x2}, l_{\ell 1}) = 1$ holds for all $\ell = x+1,\ldots,r$ and we have*

$$w_{x2}^2 \;=\; \prod_{j=x+1}^{r} l_{j1}, \qquad l_{x2} \;=\; \beta.$$

(iii) *If in* (ii) *additionally $\alpha = 0$ holds, then we have $w_{x2}^1 \;=\; -\frac{l_{x1}}{l_{x2}}$.*

(iv) *If in* (iii) *additionally $x = 0$ holds, then* $\mathrm{cone}(e_{01}, e_{02}, e_{11}, e_{21}, \ldots, e_{r-1,1})$ *is not a relevant face.*

Proof. Recall that all relations g_i of $R(A,P)$ are homogeneous of the same degree in $\mathrm{Cl}(X) = \mathbb{Z}^2$. This means that $l_{x+1,1}w_{x+1,1} = \ldots = l_{r1}w_{r1}$ holds. Since the class group of X is torsion-free, [**36**, Thm. 1.1] implies that the exponents l_{i1}, $x+1 \leq i \leq r$, are pairwise coprime. Together, this proves (i). For (ii), note that the homogeneity of the relations g_i yields

$$l_{x2}w_{x2}^2 \;=\; \beta \prod_{\ell=x+1}^{r} l_{\ell 1}. \qquad (\star)$$

Furthermore, the torsion-freeness of $\mathrm{Cl}(X)$ and Theorem 3.4.2.3 (iv) of **[3]** show that $l_{\ell 1}$ and $\gcd(l_{x1}, l_{x2})$ are coprime for all $x+1 \le \ell \le r$. Thus, $l_{x1} = -l_{x2}w_{x2}^1 - l_{\ell 1}w_{r2}^1$ implies that l_{x2} and $l_{\ell 1}$ are coprime for all $x+1 \le \ell \le r$. Together with ($\star$) we obtain

$$\delta \prod_{\ell=x+1}^{r} l_{\ell 1} \ = \ w_{x2}^2 \qquad \text{and} \qquad \delta\, l_{x2} \ = \ \beta$$

for some $\delta \in \mathbb{Z}$. Note that since $\tau := \mathrm{cone}(e_{x1}, e_{x2}, e_{i1};\ i = x+1, \dots, r)$ is relevant, local factoriality of X implies that $Q(\tau \cap E) \subseteq \mathbb{Z}^2$ is a spanning embedded monoid and thus

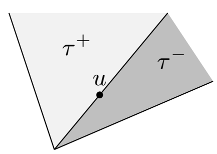
$$1 \ = \ \gcd(w_{x2}^2,\ \beta \prod_{\substack{\ell=x+1\\ \ell\neq i}}^{r} l_{\ell 1};\ \ell = x+1, \dots, r)$$

holds. We conclude $\delta = 1$, which completes the proof of (ii). Assertion (iii) is an immediate consequence of the homogeneity of the relations g_i. We turn to statement (iv). Note that $\gcd(w_{01}^2, w_{02}^2, w_{\ell 1}^2;\ \ell = 1, \dots, r-1) = l_{r1} > 1$ holds, i.e. the embedded monoid $Q(\mathrm{cone}(e_{01}, e_{02}, e_{11}, e_{21}, \dots, e_{r-1,1}) \cap E) \subseteq \mathrm{Cl}(X)$ is not spanning. Thus, local factoriality of X completes the proof. $\square$

According to Remark 1.3.3, local factoriality of $X = X(A, P, u)$ implies in particular that the effective cone $\mathrm{Eff}(X)$ is of dimension two. Since $X = X(A, P, u)$ is projective, $\mathrm{Eff}(X)$ is decomposed into two convex sets

$$\mathrm{Eff}(X) \ = \ \tau^+ \cup \tau^-$$

such that $\tau^+ \cap \tau^- = \mathrm{cone}(u)$ holds. Recall that due to $\dim(\mathrm{SAmple}(X)) = 2$ and to $u \in \mathrm{SAmple}(X)^\circ$, each of $\tau^+ \setminus \mathrm{cone}(u)$ and $\tau^- \setminus \mathrm{cone}(u)$ contains at least two of the weights w_{ij}, w_k.



Although the base point free monoid $\mathrm{BPF}(X) \subseteq \mathrm{Pic}(X)$ of a locally factorial variety $X = X(A, P, u)$ of Picard number two is in general not saturated, we obtain the following statement:

Proposition 4.6.3. *Let $X = X(A, P, u)$ be a non-toric locally factorial variety of Picard number two. Then the following are equivalent:*

(i) *X fulfills Fujita's base point free conjecture, Conjecture 4.0.1,*
(ii) *$\mathcal{K}_X + m\mathcal{L}$ is semiample for all $m \ge \dim(X) + 1$ and for all ample Weil divisor classes $\mathcal{L}$.*

Proof. By definition, a base point free Weil divisor class is semiample, thus (i) implies (ii). For the reverse direction consider $m \in \mathbb{Z}$, $m \ge \dim(X) + 1$ and denote by $\mathcal{L}$ an ample Weil divisor class. It is to show that $\mathcal{K}_X + m\mathcal{L} \in \mathrm{BPF}(X)$ holds. By Corollary 4.4.7, maximal big cones $\sigma \in \Sigma$ yield saturated embedded monoids $Q(\hat{\sigma}^* \cap E) \cap \mathrm{Pic}(X)$. This means that those monoids contain $\mathcal{K}_X + m\mathcal{L}$. It remains to show that $\mathcal{K}_X + m\mathcal{L}$ is contained in $Q(\hat{\sigma}^* \cap E)$ for all leaf cones $\sigma \in \Sigma^{\max}$. Consider a leaf cone $\sigma \in \Sigma^{\max}$. Note that it is sufficient to prove that $\mathcal{K}_X + (\dim(X)+1)\mathcal{L}$ is contained in the conductor ideal of the embedded monoid $S := Q(\hat{\sigma}^* \cap E) \subseteq \mathbb{Z}^2$. Recall that Lemma 4.1.19 gives a formula for a point $g_S \in \mathrm{Cl}(X)$ such that $(g_S + \mathrm{cone}(S)^\circ) \cap \mathbb{Z}^2 \ \subseteq \ c(\tilde{S}/S)$ holds. Thus it is sufficient to show that

$$\mathcal{K}_X + (\dim(X)+1)\mathcal{L} \ \in \ g_S + \mathrm{cone}(S)^\circ \qquad (\star)$$

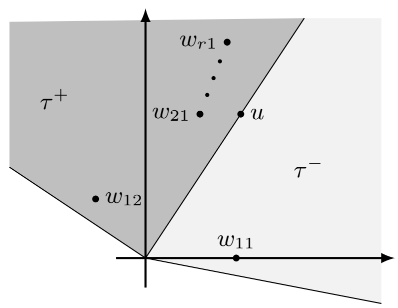
holds. After suitable admissible operations we have $\sigma \subseteq \lambda_0$ and $n_1 \ge \ldots \ge n_r$. Let

$$\nu_0 \ := \ \sharp\{e_{0j};\ e_{0j} \in \hat{\sigma}^*\} \quad \text{and} \quad \nu_\infty \ := \ \sharp\{e_k;\ e_k \in \hat{\sigma}^*\}\,.$$

Lemma 4.4.1 explains that the cone $\hat{\sigma}^*$ has exactly $\rho(X)+r-1 = r+1$ rays. Hence we obtain that $\nu_0 + n_1 + \ldots + n_r + \nu_\infty \;=\; r+1$ holds. This gives the following three cases; we will show that $(\star)$ is fulfilled in each of them:

(I) We have $\nu_0 = \nu_\infty = 0,\ n_1 = 2$ and $n_2 = \ldots = n_r = 1$.
(II) We have $\nu_0 = 1,\ \nu_\infty = 0$ and $n_1 = \ldots = n_r = 1$.
(III) We have $\nu_0 = 0,\ \nu_\infty = 1$ and $n_1 = \ldots = n_r = 1$.

Case (I). The Gale dual of the leaf cone $\sigma \in \Sigma^{\max}$ is given by $\hat{\sigma}^* = \gamma_{11,12,21,\ldots,r1}$. The homogeneity of the relations g_i yields $w_{i1} \in \operatorname{cone}(w_{r1})$ for all $i = 2,\ldots,r$. This shows in particular that $\operatorname{cone}(S) = \operatorname{cone}(w_{11}, w_{12})$ and thus $\operatorname{SAmple}(X)^\circ \subseteq \operatorname{cone}(w_{11}, w_{12})^\circ$ holds. Hence we may assume that $w_{11} \in \tau^-,\ w_{12} \in \tau^+$ and $w_{21} \in \tau^+$ hold. Furthermore, by multiplication with an unimodular (2×2)-matrix from the left, we arrive at $w_{11} \in \operatorname{cone}((1,0))$, $w_{21} = d_{21}(\alpha,\beta), \ldots, w_{r1} = d_{r1}(\alpha,\beta)$ with some $d_{i1} \in \mathbb{Z}_{\geq 1}$ and some integers $0 \leq \alpha < \beta$. The situation is as follows:



We show that $n_0 = 1$ is not possible. Assume that $n_0 = 1$ holds. This means that there is $d_{01} \in \mathbb{Z}_{\geq 1}$ such that $w_{01} = d_{01}(\alpha,\beta)$ holds. Note that local factoriality of X and $\operatorname{cone}(e_{01}, e_{11}, e_{21}, \ldots, e_{r1}) \in \operatorname{rlv}(u)$ imply that the minors of

$$\begin{bmatrix} w_{11}^1 & d_{01}\alpha & d_{21}\alpha & \ldots & d_{r1}\alpha \\ 0 & d_{01}\beta & d_{21}\beta & \ldots & d_{r1}\beta \end{bmatrix}$$

are coprime. This yields $\beta = 1 = w_{11}^1$ and $\alpha = 0$. Lemma 4.6.2 (iv) thus shows that $\operatorname{cone}(e_{01}, e_{11}e_{12}, e_{21}, \ldots, e_{r-1,1})$ is not a relevant face, which is a contradiction. Hence $n_0 \geq 2$ holds, and the homogeneity of the g_i implies that there is $0 \leq j \leq n_0$ such that $w_{0j} \in \tau^+$ holds. After suitable admissible operations we have $w_{01}, \ldots, w_{0x} \in \tau^+$ and $w_{0,x+1}, \ldots, w_{0r} \in \tau^-$ for some $1 \leq x \leq n_0$. In particular, $\gamma_{0j,11} \in \operatorname{rlv}(u)$ holds for all $1 \leq j \leq x$. Applying Remark 2.5.1 to $\gamma_{0j,11}$, we obtain $w_{0j}^2 = 1$ for all $1 \leq j \leq x$ and $w_{11}^1 = 1$. Together with Lemma 4.6.2, we thus obtain that $l_{12} = \beta,\ \gcd(l_{12},\, w_{12}^2) = 1$ and

$$Q \;=\; \left[\begin{array}{c|cc|c|c|c||c} * \cdots * \; * \cdots * & 1 & w_{12}^1 & \alpha \prod\limits_{\substack{\ell=2\\ \ell\neq 2}}^{r} l_{\ell 1} & \cdots & \alpha \prod\limits_{\substack{\ell=2\\ \ell\neq r}}^{r} l_{\ell 1} & * \cdots * \\ 1 \cdots 1 \; * \cdots * & 0 & \prod\limits_{\ell=2}^{r} l_{\ell 1} & l_{12} \prod\limits_{\substack{\ell=2\\ \ell\neq 2}}^{r} l_{\ell 1} & \cdots & l_{12} \prod\limits_{\substack{\ell=2\\ \ell\neq r}}^{r} l_{\ell 1} & * \cdots * \end{array}\right]. \qquad \text{(I.1)}$$

We now compute g_S explicitly. Since $\hat{\sigma}^* = \gamma_{11,12,21,\ldots,r1}$ holds, S is generated by $w_{11}, w_{12}, w_{21}, \ldots, w_{r1}$. Note that we have $w_{21}, \ldots, w_{r1} \in \operatorname{cone}(w_{11}, w_{12})$. In the notation of Setting 4.1.17, we thus obtain

$$
\begin{aligned}
D_1 &= \det(w_{11}, w_{12}) \;=\; \prod_{\ell=2}^{r} l_{\ell 1}\,, \\
D_2 &= \gcd\left(D_1,\;\; l_{12} \prod_{\substack{\ell=2 \\ \ell\neq 2}}^{r} l_{\ell 1},\;\; \prod_{\substack{\ell=2 \\ \ell\neq 2}}^{r} l_{\ell 1}\big(l_{12} w_{12}^1 - \alpha \prod_{\ell=2}^{r} l_{\ell 1}\big) \right) \\
&= \gcd\left(D_1,\;\; l_{12} \prod_{\substack{\ell=2 \\ \ell\neq 2}}^{r} l_{\ell 1},\;\; (-l_{11}) \prod_{\substack{\ell=2 \\ \ell\neq 2}}^{r} l_{\ell 1} \right) \\
&= \gcd(l_{21}, l_{12}, l_{11}) \prod_{\ell=3}^{r} l_{\ell 1}\,.
\end{aligned}
$$

According to Lemma 4.6.2 (ii), the integers l_{12} and l_{21} are coprime. Hence we conclude that $D_2 = \prod_{\ell=3}^{r} l_{\ell 1}$ holds. Analogously, we obtain $D_i = \prod_{\ell=i+1}^{r} l_{\ell 1}$ for all $1 \le i \le r-1$ and $D_r = 1$. We thus arrive at

$$
\begin{aligned}
g_S &= \sum_{j=2}^{r} \left(\frac{D_{j-1}}{D_j} - 1 \right) w_{j1} - \sum_{j=1}^{2} w_{1j} \\
&= (l_{21} - 1) w_{21} + \ldots (l_{r1} - 1) w_{r1} - w_{11} - w_{12} \\
&= (r-1)\deg(g_0) - \sum_{i=2}^{r} w_{i1} - w_{11} - w_{12}\,.
\end{aligned}
$$

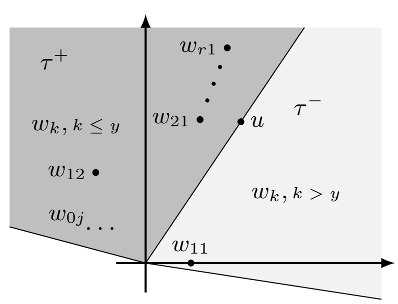
Recall that $\mathcal{K}_X = (r-1)\deg(g_0) - \sum w_{ij} - \sum w_k$ holds. By subtracting g_S in $(\star)$ we see that in order to complete the proof in Case (I), it is sufficient to show that

$$
P_I(\mathcal{L}) \;:=\; -\sum_{j=1}^{n_0} w_{0j} - \sum_{k=1}^{m} w_k + (\dim(X)+1)\mathcal{L} \;\in\; \mathrm{cone}(w_{11}, w_{12})^\circ \qquad (\star_I)
$$

holds for all $\mathcal{L} \in \mathrm{Ample}(X) \cap \mathrm{Cl}(X)$. We divide Case (I) in the following three subcases:

(I)(a) We have $w_{0j} \in \tau^+$ for all $j = 1, \ldots, n_0$.
(I)(b) We have $n_0 = 2$, $w_{01} \in \tau^+$ and $w_{02} \in \tau^-$.
(I)(c) We have $n_0 \ge 3$, $w_{01}, \ldots, w_{0x} \in \tau^+$ and $w_{0,x+1}, \ldots, w_{0n_0} \in \tau^-$ for some $1 \le x < n_0$.

In (I)(a), we have $w_{0j} \in \tau^+$ for all $j = 1, \ldots, n_0$. Since $u \in \mathrm{Mov}(R)^\circ$ holds, suitable renumbering of weights yields $w_1, \ldots, w_y \in \tau^+$ and $w_{y+1}, \ldots, w_m \in \tau^-$ for some $0 \le y \le m-1$, i.e. the situation is as follows:



Consider an index $y+1 \le k \le m$. Since $\gamma_{12,k},\ \gamma_{0j,k} \in \mathrm{rlv}(u)$ holds, Remark 2.5.1 yields

$$
1 \;=\; w_k^1 \prod_{\ell=2}^{r} l_{\ell 1} - w_k^2 w_{12}^1 \qquad \text{and} \qquad 1 \;=\; w_k^1 - w_k^2 w_{0j}^1\,.
$$

The second equation shows in particular that $w_{01}^1 = \ldots = w_{0n_0}^1 = (w_k^1 - 1)/w_k^2$ or $w_k^2 = 0$, $w_k^1 = 1$ holds for all $k > y$. Note that the latter is not possible since then the first of the above equations would yield $1 = \prod_{\ell=2}^r l_{\ell 1}$ which contradicts $l_{\ell 1} > 2$, $\ell = 2, \ldots, r$. Thus, we have $w_{01}^1 = \ldots = w_{0n_0}^1 = (w_k^1 - 1)/w_k^2$ for all $k > y$. Together with the homogeneity of g_0, this yields $w_{0j} \in \mathrm{cone}(w_{21})$. Since we have $w_{0j}^2 = 1$ and $0 \le \alpha < \beta$, we obtain $0 = \alpha = w_{0j}^1$. The homogeneity of the relation g_0 yields $w_{12} = -l_{11}/l_{12}$ and the above equations show that $w_k^1 = 1$ and

$$w_k^2 \;=\; \delta \;:=\; \frac{(1 - w_{12}^2)\, l_{12}}{l_{11}}$$

hold. In particular, we have $w_k^2 < 0$. Now consider $1 \le k \le y$. Since $\gamma_{11,k}$ and γ_{km} are relevant faces, Remark 2.5.1 yields $w_k^2 = 1$ and $1 = 1 - \delta w_k^1$, i.e. $w_k^1 = 0$ holds. We thus arrive at

$$Q \;=\; \left[\begin{array}{c|cc|c|c|c||cc} 0 \cdots 0 & 1 & -\frac{l_{11}}{l_{12}} & 0 & \cdots & 0 & 0 \cdots 0 & 1 \cdots 1 \\ 1 \cdots 1 & 0 & \prod\limits_{\ell=2}^{r} l_{\ell 1} & l_{12} \prod\limits_{\substack{\ell=2 \\ \ell \neq 2}}^{r} l_{\ell 1} & \cdots & l_{12} \prod\limits_{\substack{\ell=2 \\ \ell \neq r}}^{r} l_{\ell 1} & 1 \cdots 1 & \delta \cdots \delta \end{array}\right].$$

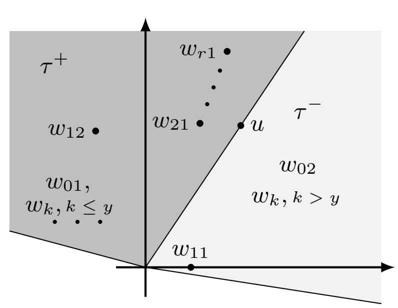
Note that $\gamma_{01,11} \in \mathrm{rlv}(u)$ holds. We conclude that we have $\mathrm{SAmple}(X) = \mathbb{Q}^2_{\ge 0}$. In order to prove $(\star_I)$, it is thus sufficient to show that $P_I((1,1))$ is contained in $\mathrm{cone}(w_{11}, w_{12})^\circ$. Since δ is strictly negative and $\dim(X) + 1 = n_0 + m + 1$ holds, we obtain

$$\begin{aligned} P_I((1,1)) \;&=\; -\sum_{j=1}^{n_0} w_{0j} - \sum_{k=1}^{m} w_k + (\dim(X)+1)(1,1) \\ &=\; n_0 \left(\begin{pmatrix} 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) + (m-y) \left(\begin{pmatrix} -1 \\ -\delta \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \\ &\quad\; + y \left(\begin{pmatrix} 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &\in\; (\mathbb{Q}^2_{\ge 0})^\circ. \end{aligned}$$

Note that $(\mathbb{Q}^2_{\ge 0})^\circ \subseteq \mathrm{cone}(w_{11}, w_{12})^\circ$ holds, i.e. the above computation completes the proof in case (I)(a).

We turn to (I)(b). We have $n_0 = 2$, $w_{01} \in \tau^+$ and $w_{02} \in \tau^-$. After suitable renumbering of variables, there is $0 \le y \le m$ such that $w_1, \ldots, w_y \in \tau^+$ and $w_{y+1}, \ldots, w_m \in \tau^-$ hold. Applying Remark 2.5.1 to $\gamma_{01,11}$ and to $\gamma_{11,k}$, $k \le y$, yields the following:

$$Q \;=\; \left[\begin{array}{cc|cc|c|c|c||cc} * & * & 1 & w_{12}^1 & \alpha \prod\limits_{\substack{\ell=2 \\ \ell \neq 2}}^{r} l_{\ell 1} & \cdots & \alpha \prod\limits_{\substack{\ell=2 \\ \ell \neq r}}^{r} l_{\ell 1} & * \cdots * & * \cdots * \\ 1 & * & 0 & \prod\limits_{\ell=2}^{r} l_{\ell 1} & l_{12} \prod\limits_{\substack{\ell=2 \\ \ell \neq 2}}^{r} l_{\ell 1} & \cdots & l_{12} \prod\limits_{\substack{\ell=2 \\ \ell \neq r}}^{r} l_{\ell 1} & 1 \cdots 1 & * \cdots * \end{array}\right].$$



Because of $\gamma_{02,12} \in \mathrm{rlv}(u)$, Remark 2.5.1 yields

$$1 \;=\; w_{02}^1 \prod_{\ell=2}^{r} l_{\ell 1} - w_{02}^2 w_{12}^1 \,.$$

Since w_{02}^1 is strictly positive, we obtain that w_{12}^1 and w_{02}^2 have the same algebraic sign. Furthermore, the first summand of the right-hand side is strictly greater than one. Thus, $w_{12}^1, w_{02}^2 \neq 0$ and

$$w_{02}^2 \;=\; \frac{w_{02}^1 \prod_{\ell=2}^{r} l_{\ell 1} - 1}{w_{12}^1}$$

hold. Note that $\dim(X) + 1 = m + n_0 + 1$ and $\mathrm{SAmple}(X) \subseteq \mathrm{cone}(w_{11}, w_{12})$ hold, i.e. in order to prove $(\star_I)$ in Case (I)(b), it is enough to show that

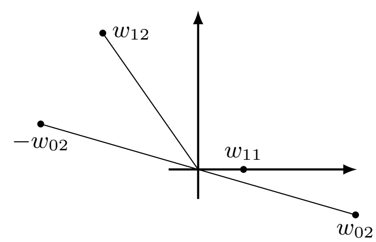
$$-w_{01} + \mathcal{L},\; -w_{02} + \mathcal{L},\; -w_k + \mathcal{L} \;\in\; \mathrm{cone}(w_{11}, w_{12}) \quad (\star_{Ib})$$

holds for all $\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2) \in \mathrm{Ample}(X) \cap \mathrm{Cl}(X)$. We will show the claims of $(\star_{Ib})$ in the two cases $w_{12}^1 < 0$ and $w_{12}^1 > 0$.

First assume that $w_{12}^1 < 0$ holds. We consider $-\tilde{w} + \mathcal{L}$ with $\tilde{w} = (\tilde{w}_1, \tilde{w}_2) \in \{w_{01}, w_k;\; 1 \leq k \leq y\}$. Note that we have $\tilde{w}_2 = 1$ and $\mathcal{L}_2 > 0$, i.e. $-\tilde{w}_2 + \mathcal{L}_2$ is greater than or equal to zero. Furthermore, we have $\tilde{w}_1 < 0$. Thus, $\mathcal{L} \in \mathrm{cone}(\tilde{w}, w_{11})^\circ \cap \mathrm{cone}(w_{02}, w_{12})^\circ$ shows that

$$-\tilde{w} + \mathcal{L} \;\in\; \kappa := \mathrm{cone}(w_{11}, w_{12})$$

holds, which proves $(\star_{Ib})$ for $\tilde{w}$. For $-w_{02}$, the situation is as follows:



We see that $-w_{02} + \mathcal{L} \in \mathrm{cone}(-w_{02}, w_{11})^\circ$ holds. Thus we have

$$-w_{02} + \mathcal{L} \;\in\; \mathrm{cone}(-w_{02}, w_{12})^\circ \;\cup\; \mathrm{cone}(w_{11}, w_{12})\,.$$

Remark 2.5.1 applied to $\gamma_{02,12}$ shows that $(-w_{02}, w_{12})$ is a lattice basis for $\mathbb{Z}^2$. If $-w_{02} + \mathcal{L} \in \mathrm{cone}(-w_{02}, w_{12})^\circ$ held, there would be $a, b \in \mathbb{Z}_{>0}$ such that $-w_{02} + \mathcal{L}$ equals $a(-w_{02}) + bw_{12}$. This would yield

$$\mathcal{L} \;=\; (a-1)(-w_{02}) + bw_{12} \;\in\; \mathrm{cone}(-w_{02}, w_{12})\,,$$

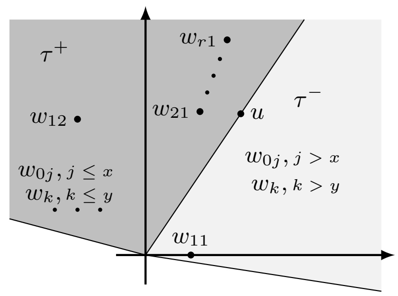
which contradicts the ampleness of $\mathcal{L}$. Hence we conclude $-w_{02} + \mathcal{L} \in \mathrm{cone}(w_{11}, w_{12})$. Analogously, we see that $-w_k + \mathcal{L} \in \mathrm{cone}(w_{11}, w_{12})$ holds for all $y + 1 \leq k \leq m$.

We now turn to the case $w_{12}^1 > 0$. Since $0 \leq \alpha < l_{12}$, $w_{01}^2 = 1$ and $w_{21} \in \mathrm{cone}(w_{01}, w_{02})^\circ$ hold, we obtain $w_{01}^1 \leq 0$. This shows that $-w_{01} + \mathcal{L}$ is contained in $\mathrm{cone}(w_{11}, w_{12})$. Remark 2.5.1 applied to $\gamma_{02,12}$ shows that (w_{02}, w_{12}) is a lattice basis for $\mathbb{Z}^2$. Since $\mathrm{SAmple}(X) \subseteq \mathrm{cone}(w_{02}, w_{12})$ holds, each ample class $\mathcal{L}$ has a representation $\mathcal{L} = \alpha w_{02} + \beta w_{12}$ with integers $\alpha, \beta \in \mathbb{Z}_{>0}$. This shows that $-w_{02} + \mathcal{L}$ is contained in $\mathrm{cone}(w_{02}, w_{12})$. We showed above that w_{02}^2 has the same algebraic sign as w_{12}^1, which implies in particular that $-w_{02} + \mathcal{L}$ is contained in $\mathrm{cone}(w_{11}, w_{12})$. Consider an index $1 \leq k \leq y$. Remark 2.5.1 applied to $\gamma_{02,k}$ yields $1 = w_{02}^1 - w_{02}^2 w_k^1$ $(*)$, i.e. $w_k^1 \geq 0$ holds. Note that $w_{12}^2 / w_{12}^1 > w_{21}^2 / w_{21}^1 > 1$ holds. Since (w_{02}, w_{12}) is a lattice basis for $\mathbb{Z}^2$, we conclude that $w_{02}^2 / w_{02}^1 > 1$ holds. Together with $(*)$, this gives

$$w_k^1 \;<\; \frac{w_{02}^2}{w_{02}^1} w_k^1 \;=\; 1 - \frac{1}{w_{02}^1} \;<\; 1\,.$$

We conclude that $w_k^1 = 0$ holds. Since $\mathcal{L}_2$ is strictly positive, we obtain that $-w_k + \mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2 - 1)$ is contained in $\operatorname{cone}(w_{11}, w_{12})$. Now consider a weight $w_k \in \tau^-$. Remark 2.5.1 applied to $\gamma_{12,k}$ shows that $w_k^2 > 0$ holds and that (w_k, w_{12}) is a lattice basis for $\mathbb{Z}^2$. Since $\operatorname{SAmple}(X) \subseteq \operatorname{cone}(w_k, w_{12})$ holds, each ample class $\mathcal{L}$ has a representation $\mathcal{L} = \alpha w_k + \beta w_{12}$ with integers $\alpha,\ \beta \in \mathbb{Z}_{>0}$. This shows that $-w_k + \mathcal{L}$ is contained in $\operatorname{cone}(w_k, w_{12})$. Because of $w_k^2 > 0$, we obtain $-w_k + \mathcal{L} \in \operatorname{cone}(w_{11}, w_{12})$. As argued above, this completes the proof in Case (I)(b).

We turn to Case (I)(c) where $n_0 \geq 3$ holds. Moreover, for some $1 \leq x < n_0$, we have $w_{01}, \ldots, w_{0x} \in \tau^+$ and $w_{0,x+1}, \ldots, w_{0n_0} \in \tau^-$. After suitable renumbering of variables, there is $0 \leq y \leq m$ such that $w_1, \ldots, w_y \in \tau^+$ and $w_{y+1}, \ldots, w_m \in \tau^-$ hold. Recall that the degree matrix is as in (I.1). Applying Remark 2.5.1 to $\gamma_{11,k}$, $k \leq y$, yields $w_k^2 = 1$ for all $1 \leq k \leq y$, i.e. the weights are arranged as follows:



Consider an index $x + 1 \leq j \leq n_0$. Since $\gamma_{0j,12},\ \gamma_{01,0j} \in \operatorname{rlv}(u)$ holds, Remark 2.5.1 yields

$$1 \ = \ w_{0j}^1 w_{12}^2 - w_{0j}^2 w_{12}^1 \text{ (i)} \qquad \text{and} \qquad 1 \ = \ w_{0j}^1 - w_{0j}^2 w_{01}^1 \text{ (ii)}.$$

Note that $w_{0j}^1 w_{12}^2 \geq 2$ holds. Thus, the first of the above equations shows in particular that w_{0j}^2 and w_{12}^1 are non-zero and have the same algebraic sign. By inserting (i) into (ii), we obtain $1 - w_{12}^2 = w_{0j}^2 (w_{01}^1 w_{12}^2 - w_{12}^1)$. Note that since $w_{12}^2 = \prod_{\ell=2}^r l_{\ell 1} \geq 2$ and thus $1 - w_{12}^2 < 0$ holds, we have $w_{01}^1 w_{12}^2 - w_{12}^1 \neq 0$. We conclude that

$$\delta_2 \ := \ w_{0j}^2 \ = \ \frac{1 - w_{12}^2}{w_{01}^1 w_{12}^2 - w_{12}^1} \qquad \text{(II.1)}$$

holds. Together with the second of the above equations, we arrive at

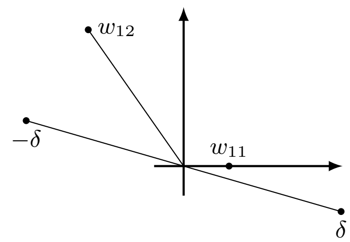
$$\delta_1 \ := \ w_{0j}^1 \ = \ \frac{w_{01}^1 - w_{12}^1}{w_{01}^1 w_{12}^2 - w_{12}^1}. \qquad \text{(II.2)}$$

In particular, we have $\delta_1 = w_{0j}^1 > 0$ since w_{0j} is contained in τ^-. Analogously, we apply Remark 2.5.1 to $\gamma_{01,k}, \gamma_{12,k} \in \operatorname{rlv}(u)$ for all $y + 1 \leq k \leq m$ and obtain $w_k = (\delta_1, \delta_2)$. Since $\gamma_{0n_0,0j} \in \operatorname{rlv}(u)$ and $w_{0j}^2 = 1$ hold for all $1 \leq j \leq x$, the above formulas for $w_{0n_0}^1$ and $w_{0n_0}^2$ show in particular that $w_{01} = \ldots = w_{0x}$ holds. Now consider an index $1 \leq k \leq y$, i.e. $w_k \in \tau^+$ holds. Since $\gamma_{0n_0,k} \in \operatorname{rlv}(u)$ holds, the above formulas for $w_{0n_0}^1$ and $w_{0n_0}^2$ yield $w_k = w_{01}$. We arrive at

$$[w_{ij}]_{i,j} \ = \ \left[\begin{array}{cccccc|cc|c|c|c} w_{01}^1 & \cdots & w_{01}^1 & \delta_1 & \cdots & \delta_1 & 1 & w_{12}^1 & \alpha \prod\limits_{\substack{\ell=2 \\ \ell\neq 2}}^{r} l_{\ell 1} & \ldots & \alpha \prod\limits_{\substack{\ell=2 \\ \ell\neq r}}^{r} l_{\ell 1} \\ 1 & \cdots & 1 & \delta_2 & \cdots & \delta_2 & 0 & \prod_{j=2}^r l_{j1} & l_{12} \prod\limits_{\substack{\ell=2 \\ \ell\neq 2}}^{r} l_{\ell 1} & \ldots & l_{12} \prod\limits_{\substack{\ell=2 \\ \ell\neq r}}^{r} l_{\ell 1} \end{array}\right],$$

$$[w_k]_k \ = \ \left[\begin{array}{cccccc} w_{01}^1 & \cdots & w_{01}^1 & \delta_1 & \cdots & \delta_1 \\ 1 & \cdots & 1 & \delta_2 & \cdots & \delta_2 \end{array}\right].$$

We now show that $(\star_I)$ is fulfilled in Case (I)(c). Let $\mathcal{L} \in \mathbb{Z}^2$ be an ample class. Recall that in the beginning of Case (I) we obtained $\operatorname{SAmple}(X)^\circ \subseteq \operatorname{cone}(w_{11}, w_{12})^\circ$. This proves $\mathcal{L}_2 > 0$ and $\mathcal{L} \in \operatorname{cone}(w_{11}, w_{12})^\circ$. Since $n_1 + \ldots + n_r = r + 1$ holds, we obtain $\dim(X) + 1 = n + m - r = n_0 + m + 1$. This means that in order to prove $(\star_I)$ in Case (I)(c), it is sufficient to show that $-(w_{01}^1, 1) + \mathcal{L}$ and $-(\delta_1, \delta_2) + \mathcal{L}$ are contained in $\mathrm{cone}(w_{11}, w_{12})$. First consider $(w_{01}^1, 1) \in \tau^+$. Since $w_{01}^2 = 1$ and $w_{21}^2 > w_{21}^1 \geq 0$ hold, we obtain $w_{01}^1 \leq 0$. Together with $\mathcal{L} \in \mathrm{cone}(w_{11}, w_{12})^\circ$, this yields $-(w_{01}^1, 1) + \mathcal{L} \in \mathrm{cone}(w_{11}, w_{12})$. Now we consider $\delta := (\delta_1, \delta_2) \in \tau^-$. Here we distinguish the subcases $w_{12}^1 < 0$ and $w_{12}^1 > 0$. If $w_{12}^1 < 0$ holds, then because of $w_{0j}^1 w_{12}^2 \geq 2$, equation (i) shows that $w_{0j}^2 = \delta_2 < 0$ holds. Thus, the situation is as follows:



We see that $-\delta + \mathcal{L}$ is contained in the relative interior of $\mathrm{cone}(-\delta, w_{11})$. Thus,

$$-\delta + \mathcal{L} \;\in\; \mathrm{cone}(-\delta, w_{12})^\circ \;\cup\; \mathrm{cone}(w_{11}, w_{12})$$

holds. Note that Remark 2.5.1 applied to $\gamma_{0n_0,12}$ shows that $(-\delta, w_{12})$ is a lattice basis for $\mathbb{Z}^2$. This means that if $-\delta + \mathcal{L} \in \mathrm{cone}(-\delta, w_{12})^\circ$ held, there would be integers $a, b \in \mathbb{Z}_{>0}$ such that $-\delta + \mathcal{L} = a(-\delta) + bw_{12}$ holds. But this yields

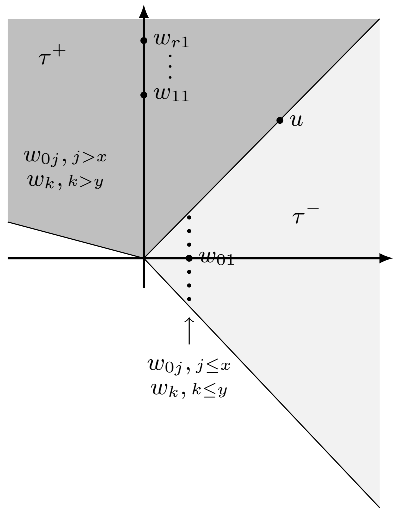
$$\mathcal{L} \;=\; (a-1)(-\delta) + bw_{12} \;\in\; \mathrm{cone}(-\delta, w_{12}),$$

which contradicts the ampleness of $\mathcal{L}$. Hence we conclude $-\delta + \mathcal{L} \in \mathrm{cone}(w_{11}, w_{12})$. Now we assume that $w_{12}^1 > 0$ holds. Together with equation (II.1), $w_{01}^1 < 0$, $w_{12}^2 > 0$ and $1 - w_{12}^2 < 0$, this yields $\delta_2 > 0$. Since $\gamma_{0n_0,12} \in \mathrm{rlv}(u)$ holds, we conclude that $\mathrm{SAmple}(X) \subseteq \mathrm{cone}(w_{12}, \delta)$ holds. Thus, each ample class $\mathcal{L}$ has a representation $\mathcal{L} = \alpha w_{12} + \beta\delta$ with integers α, $\beta \in \mathbb{Z}_{>0}$. This shows that $-\delta + \mathcal{L}$ is contained in $\mathrm{cone}(w_{12}, \delta)$. Because of $\delta_2 > 0$, this shows $-\delta + \mathcal{L} \in \mathrm{cone}(w_{11}, w_{12})$. As argued above, this completes the proof in Case (I)(c).

Case (II). We have $\nu_0 = 1$, $\nu_\infty = 0$ and $n_1 = \ldots = n_r = 1$. Thus we may assume that the Gale dual of the leaf cone $\sigma \in \Sigma^{\max}$ is given by $\hat{\sigma}^* = \gamma_{01,11,21,\ldots,r1}$. The homogeneity of the relations g_i yields $w_{i1} \in \mathrm{cone}(w_{r1})$ for all $i = 1, \ldots, r$. Since X is $\mathbb{Q}$-factorial, the cone $Q(\hat{\sigma}^*)$ is two-dimensional. We may assume that $w_{01} \in \tau^-$ and $w_{11}, \ldots,\ w_{r1} \in \tau^+$ hold. Together with the homogeneity of the g_i, this shows in particular that $n_0 \geq 2$ and $w_{0j} \in \tau^+$ holds for some $2 \leq j \leq n_0$. Thus after suitable admissible operations, $w_{0,x+1}, \ldots, w_{0n_0} \in \tau^+$ and $w_{01}, \ldots, w_{0x} \in \tau^-$ hold for some $1 \leq x \leq n_0 - 1$. Furthermore by multiplication with an unimodular (2×2)-matrix from the left, we arrive at $w_{01} \in \mathrm{cone}((1,0))$ and $w_{11}, \ldots, w_{r1} \in \mathrm{cone}((\alpha, \beta))$ for some integers α, β with $0 \leq \alpha < \beta$. Note that Lemma 4.6.2 (i) implies that

$$w_{i1}^1 \;=\; \alpha \prod_{\ell=1,\ell\neq i}^{r} l_{\ell 1} \qquad \text{and} \qquad w_{i1}^2 \;=\; \beta \prod_{\ell=1,\ell\neq i}^{r} l_{\ell 1}$$

hold for all $1 \leq i \leq r$. According to Remark 1.3.3, local factoriality of X together with $\gamma_{01,11,\ldots,r1} \in \mathrm{rlv}(u)$ yields $\beta = 1$ and $w_{01} = (1,0)$. In particular, this implies $\alpha = 0$. After suitable renumbering of variables, there is $0 \leq y \leq m$ such that $w_1, \ldots, w_y \in \tau^-$ and $w_{y+1}, \ldots, w_m \in \tau^+$ hold. Consider $2 \leq j \leq x$ and $1 \leq k \leq y$. According to Remark 1.3.3, local factoriality of X together with $\gamma_{0j,11,21,\ldots,r1}$, $\gamma_{k,11,21,\ldots,r1} \in \mathrm{rlv}(u)$ shows that $w_{0j}^1 = w_k^1 = 1$ holds. The homogeneity of the relations g_i together with $\deg(g_i)^1 = 0$ and $w_{01}^1 > 0$ yields $w_{0j}^1 < 0$ for some $x + 1 \leq j \leq n_0$; say $w_{0n_0}^1 < 0$. The situation is as follows:



$$[w_{ij}]_{ij} \;=\; \left[\begin{array}{ccccccc|c|c|c} 1 & 1 & \cdots & 1 & w^1_{0x+1} & \cdots & w^1_{0n_0} & 0 & \cdots & 0 \\ 0 & w^2_{02} & \cdots & w^2_{0x} & w^2_{0x+1} & \cdots & w^2_{0n_0} & \prod\limits_{\substack{\ell=1\\ \ell\neq 1}}^{r} l_{\ell 1} & \cdots & \prod\limits_{\substack{\ell=1\\ \ell\neq r}}^{r} l_{\ell 1} \end{array}\right],$$

$$[w_k]_k \;=\; \left[\begin{array}{cccccc} 1 & \cdots & 1 & w^1_{y+1} & \cdots & w^1_m \\ w^2_1 & \cdots & w^2_y & w^2_{y+1} & \cdots & w^2_m \end{array}\right].$$

We now compute g_S explicitly. Since $\hat{\sigma}^* = \gamma_{01,11,21,\ldots,r1}$ holds, $S = Q(\hat{\sigma}^* \cap E)$ is generated by $w_{01}, w_{11}, w_{21}, \ldots, w_{r1}$. In the notation of Setting 4.1.17, we thus obtain

$$D_1 \;=\; \prod_{\ell=2}^{r} l_{\ell 1} \qquad \text{and} \qquad D_2 \;=\; \gcd\left(D_1,\; \prod_{\substack{\ell=1\\ \ell\neq 2}}^{r} l_{\ell 1},\; 0\right) \;=\; \prod_{\ell=3}^{r} l_{\ell 1}.$$

Analogously, we obtain $D_i = \prod_{\ell=i+1}^{r} l_{\ell 1}$ for all $i = 1, \ldots, r-1$ and $D_r = 1$. We arrive at

$$\begin{aligned} g_S \;&=\; \sum_{j=2}^{r} \left(\frac{D_{j-1}}{D_j} - 1\right) w_{j1} - w_{01} - w_{11} \\ &=\; (l_{21}-1)w_{21} + \ldots (l_{r1}-1)w_{r1} - w_{01} - w_{11} \\ &=\; (r-1)\deg(g_0) - \sum_{i=1}^{r} w_{i1} - w_{01}. \end{aligned}$$

Recall that $\mathcal{K}_X = (r-1)\deg(g_0) - \sum w_{ij} - \sum w_k$ and $Q(\hat{\sigma}^*) = \mathbb{Q}^2_{\geq 0}$ hold. By subtracting g_S in $(\star)$ we see that in order to complete the proof in Case (II), it is sufficient to show that

$$P_{II}(\mathcal{L}) \;:=\; -\sum_{j=2}^{n_0} w_{0j} - \sum_{k=1}^{m} w_k + (\dim(X)+1)\mathcal{L} \;\in\; \mathbb{Q}^2_{>0} \qquad (\star_{II})$$

holds for all $\mathcal{L} \in \mathrm{Ample}(X) \cap \mathrm{Cl}(X)$. We first show that $n_0 \geq 3$ holds.

Indeed, assume that $n_0 = 2$ holds. In this case, homogeneity of g_0 yields

$$l_{01} + l_{02}w^1_{02} \;=\; 0 \qquad \text{and} \qquad l_{02}w^2_{02} \;=\; \prod_{\ell=1}^{r} l_{\ell 1}.$$

Since $\gcd(l_{01}, l_{02})$ and $l_{\ell 1}$ are coprime for all $\ell = 1, \ldots, r$, we conclude that $l_{02} = 1$, $w_{02}^2 = l_{11} \cdots l_{r1}$ and $w_{02}^1 = -l_{01}$ hold. Thus the degree matrix Q is given by

$$Q = \left[\begin{array}{cc|c|c|c||cccccc} 1 & -l_{01} & 0 & \cdots & 0 & 1 & \cdots & 1 & * & \cdots & * \\ 0 & \prod_{\ell=1}^{r} l_{\ell 1} & \prod_{\substack{\ell=1\\ \ell\neq 1}}^{r} l_{\ell 1} & \cdots & \prod_{\substack{\ell=1\\ \ell\neq r}}^{r} l_{\ell 1} & * & \cdots & * & * & \cdots & * \end{array}\right].$$

Note that $Q(\gamma_{01,02,21,31,\ldots,r1})^\circ$ contains $Q(\hat{\sigma}^*)^\circ$. Thus the cone $\gamma_{01,02,21,31,\ldots,r1}$ is a relevant face. But the embedded monoid $Q(\gamma_{01,02,21,31,\ldots,r1}) \subseteq \mathbb{Z}^2$ is not spanning because of $l_{11} = \gcd(w_{01}^2, w_{02}^2, w_{21}^2, \ldots, w_{r1}^2)$. This contradicts Remark 1.3.3 since X is locally factorial. Thus $n_0 = 2$ is not possible and $n_0 \geq 3$ holds.

Remark 2.5.1 applied to $\gamma_{01,0j}$, $j > x$, and to $\gamma_{01,k}$, $k > y$, implies $w_{0j}^2 = w_k^2 = 1$ for all $j > x$, $k > y$. Applying again Remark 2.5.1, this time to $\gamma_{0j,0n_0}$, $j \leq x$, and to $\gamma_{0n_0,k}$, $k \leq y$, yields $1 = 1 - w_{0j}^2 w_{0n_0}^1$ and $1 = 1 - w_k^2 w_{0n_0}^1$, respectively. Since $w_{0n_0}^1 < 0$ holds, this yields $w_{0j}^2 = w_k^2 = 0$ for all $j \leq x$, $k \leq y$. Thus, we obtain the degree matrix

$$Q = \left[\begin{array}{cccccc|c|c|c||cccccc} 1 & \cdots & 1 & * & \cdots & * & 0 & \cdots & 0 & 1 & \cdots & 1 & * & \cdots & * \\ 0 & \cdots & 0 & 1 & \cdots & 1 & \prod_{\substack{\ell=1\\ \ell\neq 1}}^{r} l_{\ell 1} & \cdots & \prod_{\substack{\ell=1\\ \ell\neq r}}^{r} l_{\ell 1} & 0 & \cdots & 0 & 1 & \cdots & 1 \end{array}\right].$$

Set $\mu := \max(0,\, w_{0j}^1,\, w_k^1;\;\, x+1 \leq j \leq n_0,\, y+1 \leq k \leq m)$. Then $\mathrm{SAmple}(X) = \mathrm{cone}((1,0),(\mu,1))$ holds. We consider the ample class $\mathcal{L}_\mu := (\mu+1, 1)$. Note that in order to prove $(\star_{II})$, it is sufficient to show that $P_{II}(\mathcal{L}_\mu)$ is contained in $\mathbb{Q}^2_{\geq 0}$. Since $n_1 + \ldots + n_r = r$ holds, we obtain $\dim(X) + 1 = n + m - r = n_0 + m$. Hence it is sufficient to show that for all $w \in \{w_{0j},\, w_k;\; 2 \leq j \leq n_0,\, 1 \leq k \leq m\}$, the Weil divisor class $-w + \mathcal{L}_\mu$ is contained in $\mathbb{Q}^2_{\geq 0}$. First consider $w_{0j},\, w_k \in \tau^-$. Recall that this means that $w_{0j} = w_k = (1,0)$ holds. We obtain

$$-w_{0j} + \mathcal{L}_\mu \;=\; -w_k + \mathcal{L}_\mu \;=\; (\mu, 1) \;\in\; \mathbb{Q}^2_{\geq 0}\,.$$

Now let $w_{0j},\, w_k \in \tau^+$. We showed above that $w_{0j}^2 = w_k^2 = 1$ holds and by definition of μ, we have $\mu \geq w_{0j}^2, w_k^2$. We conclude

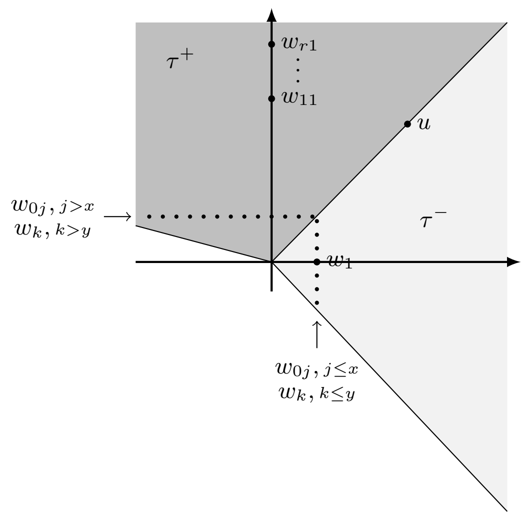
$$-w_{0j} + \mathcal{L}_\mu = (-w_{0j}^1 + \mu + 1, 0), \quad -w_k + \mathcal{L}_\mu = (-w_k^1 + \mu + 1, 0) \;\in\; \mathbb{Q}^2_{\geq 0}\,.$$

As argued above, this completes the proof in Case (II).

Case (III). We have $\nu_0 = 0$, $\nu_\infty = 1$ and $n_1 = \ldots = n_r = 1$. Thus we may assume that the Gale dual of the leaf cone $\sigma \in \Sigma^{\max}$ is given by $\hat{\sigma}^* = \gamma_{11,21,\ldots,r1,1}$. The homogeneity of the relations g_i yields $w_{i1} \in \mathrm{cone}(w_{r1})$ for all $i = 1, \ldots, r$. Since X is $\mathbb{Q}$-factorial, the cone $Q(\hat{\sigma}^*)$ is two-dimensional. We may assume that $w_{01} \in \tau^-$ and $w_{11}, \ldots,\, w_{r1} \in \tau^+$ hold. Furthermore by multiplication with an unimodular (2×2)-matrix from the left, we arrive at $w_{01} \in \mathrm{cone}((1,0))$ and $w_{11}, \ldots, w_{r1} \in \mathrm{cone}((\alpha,\beta))$ for some integers $0 \leq \alpha < \beta$. Note that Lemma 4.6.2 (i) implies that we have

$$w_{i1}^1 = \alpha \prod_{\ell=1, \ell\neq i}^{r} l_{\ell 1} \qquad \text{and} \qquad w_{i1}^2 = \beta \prod_{\ell=1,\ell\neq i}^{r} l_{\ell 1}, \quad 1 \leq i \leq r\,.$$

According to Remark 1.3.3, local factoriality of X together with $\gamma_{01,11,\ldots,r1} \in \mathrm{rlv}(u)$ shows $\beta = 1$ and $w_1 = (1,0)$. This implies in particular, that $\alpha = 0$ holds. Since the relations g_i are homogeneous of degree $l_{11}w_{11}$, there is some $w_{0j} \in \tau^+$, i.e. we may assume that $w_{01}, \ldots, w_{0x} \in \tau^-$ and $w_{0,x+1}, \ldots, w_{0n_0} \in \tau^+$ hold for some $0 \leq x < n_0$. After suitable renumbering of variables, there is $0 \leq y \leq m$ with $w_1, \ldots, w_y \in \tau^-$ and $w_{y+1}, \ldots, w_m \in \tau^+$. Note that since $\gcd(l_{0j};\; 1 \leq j \leq n_0)$ and $l_{11} \cdots l_{r1} = \deg(g)$ are coprime, we conclude that $n_0 \geq 2$ holds. For all $x+1 \leq j \leq n_0$ and $y+1 \leq k \leq m$, Remark 2.5.1 applied to $\gamma_{0j,1}$, $\gamma_{1,k}$ yields $w_{0j}^2 = w_k^2 = 1$. Since the cone $\mathrm{SAmple}(X)$ is contained in the relative interior of the moving cone of X, we have $x \geq 1$ or $y \geq 2$. For all $1 \leq j \leq x$, $1 \leq k \leq y$, Remark 2.5.1 applied to $\gamma_{0j,11,\ldots,r1}$, $\gamma_{11,\ldots,r1,k}$ yields $w_{0j}^1 = w_k^1 = 1$. The degree matrix is as below and the arrangement of weights is as follows:



$$Q \;=\; \left[\begin{array}{ccccccc|c|c|c||cccccc} 1 & \cdots & 1 & * & \cdots & * & & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & * & \cdots & * \\ * & \cdots & * & 1 & \cdots & 1 & & \prod\limits_{\substack{\ell=1\\ \ell\neq 1}}^{r} l_{\ell 1} & \cdots & \prod\limits_{\substack{\ell=1\\ \ell\neq r}}^{r} l_{\ell 1} & 0 & * & \cdots & * & 1 & \cdots & 1 \end{array}\right].$$

Note that the grading of $R(A,P)$ is pointed. Thus we have

$$w_{0j}^2, w_k^2 \geq 0 \text{ for all } j \leq x,\, k \leq y \quad \text{or} \quad w_{0j}^1, w_k^1 \geq 0 \text{ for all } j > x,\, k > y\,.$$

We now compute g_S explicitly. Since $\hat{\sigma}^* = \gamma_{11,21,\ldots,r1,1}$ holds, $S = Q(\hat{\sigma}^* \cap E)$ is generated by $w_{11}, w_{21}, \ldots, w_{r1}, w_1$. In the notation of Setting 4.1.17, we thus obtain

$$D_1 \;=\; \prod_{\ell=2}^{r} l_{\ell 1} \qquad \text{and} \qquad D_2 \;=\; \gcd\Bigg(D_1,\; \prod_{\substack{\ell=1\\ \ell\neq 2}}^{r} l_{\ell 1},\;\; 0\Bigg) \;=\; \prod_{\ell=3}^{r} l_{\ell 1}\,.$$

Analogously, we obtain $D_i = \prod_{\ell=i+1}^{r} l_{\ell 1}$ for all $i = 1, \ldots, r-1$ as well as $D_r = 1$. We arrive at

$$\begin{aligned} g_S \;&=\; \sum_{j=2}^{r}\left(\frac{D_{j-1}}{D_j} - 1\right) w_{j1} - w_{11} - w_1 \\ &=\; (l_{21}-1)w_{21} + \ldots (l_{r1}-1)w_{r1} - w_{11} - w_1 \\ &=\; (r-1)\deg(g_0) - \sum_{i=1}^{r} w_{i1} - w_1\,. \end{aligned}$$

Recall that $\mathcal{K}_X = (r-1)\deg(g_0) - \sum w_{ij} - \sum w_k$ and $Q(\hat{\sigma}^*) = \mathbb{Q}^2_{\geq 0}$ hold. Furthermore, we have $\dim(X) + 1 = n + m - r = n_0 + m$. By subtracting g_S in $(\star)$ we see that in order to complete the proof in Case (III), it is sufficient to show that

$$P_{III}(\mathcal{L}) \;:=\; -\sum_{j=1}^{n_0} w_{0j} - \sum_{k=2}^{m} w_k + (n_0+m)\mathcal{L} \;\in\; \mathbb{Q}^2_{>0} \qquad (\star_{III})$$

holds for all $\mathcal{L} \in \mathrm{Ample}(X) \cap \mathrm{Cl}(X)$.

We show that $n_0 = 2$ together with $w_{01} \in \tau^-$ and $w_{02} \in \tau^+$ is not possible. Assume that $n_0 = 2$, $w_{01} \in \tau^-$ and $w_{02} \in \tau^+$ holds. In this case, homogeneity of the relation g_0 yields

$$l_{01} + l_{02}w_{02}^1 = 0 \qquad \text{and} \qquad l_{01}w_{01}^2 + l_{02} = \prod_{\ell=1}^{r} l_{\ell 1}\,.$$

Since $\gcd(l_{01}, l_{02})$ and $l_{\ell 1}$ are coprime for all $1 \le \ell \le r$, inserting the first into the second equation shows that $l_{02} = 1$ and $(1 - w_{02}^1 w_{01}^2) = l_{11} \cdots l_{r1}$ hold. Thus the minors of the $(2 \times \mathrm{r})$-matrix $(w_{01}, w_{02}, w_{21}, w_{31}, \ldots, w_r)$ are divided by $l_{\ell 1}$. Note that $l_{\ell 1}$ is at least two, which implies that the embedded monoid $Q(\gamma_{01,02,21,31,\ldots,r1} \cap E) \subseteq \mathbb{Z}^2$ is not spanning. Since $\gamma_{01,02,21,31,\ldots,r1}$ is a relevant face, this contradicts local factoriality of X.

Hence we have $n_0 \ge 3$ or $n_0 = 2$, $w_{01}, w_{02} \in \tau^+$. Remark 2.5.1 applied to the cones of the form $\gamma_{0j_1,0j_2}$, $\gamma_{0j,k}$ and γ_{k_1,k_2} that are relevant faces shows that we are in one of the following situations:

(a) $w_{0j}^2 = w_k^2 = 0$ for all $1 \le j \le x$, $1 \le k \le y$,
(b) $w_{0j}^1 = w_k^1 = 0$ for all $x+1 \le j \le n_0$, $y+1 \le k \le m$.

Note that the semiample cone of X is given by $\mathrm{SAmple}(X) = \mathrm{cone}((\mu_a, 1), (1, \mu_b))$, where we set

$$\mu_a := \max(w_{0j}^1,\, w_k^1,\, 0;\; x+1 \le j \le n_0,\, y+1 \le k \le m)\,,$$

$$\mu_b := \max(w_{0j}^2,\, w_k^2,\, 0;\; 1 \le j \le x,\, 1 \le k \le y)\,.$$

In Case (III)(a), we consider the ample class $\mathcal{L}_a := (\mu_a + 1, 1)$. Note that in order to prove $(\star_{III})$ in Case (III)(a), it is sufficient to show that $P_{III}(\mathcal{L}_a)$ is contained in $\mathbb{Q}^2_{>0}$. Hence it is sufficient to show that for all $w \in \{w_{0j},\, w_k;\; 1 \le j \le n_0,\, 2 \le k \le m\}$, the Weil divisor class $-w + \mathcal{L}_a$ is contained in $\mathbb{Q}^2_{\ge 0}$. First consider $w_{0j},\, w_k \in \tau^-$. Recall that this means that we have $w_{0j} = w_k = (1, 0)$. We obtain

$$-w_{0j} + \mathcal{L}_a \;=\; -w_k + \mathcal{L}_a \;=\; (\mu_a, 1) \;\in\; \mathbb{Q}^2_{\ge 0}\,.$$

Now let $w_{0j},\, w_k \in \tau^+$. We showed above that $w_{0j}^2 = w_k^2 = 1$ holds and by definition of μ_a, we have $\mu_a \ge w_{0j}^2, w_k^2$. We conclude

$$-w_{0j} + \mathcal{L}_a = (-w_{0j}^1 + \mu_a + 1, 0),\; -w_k + \mathcal{L}_a = (-w_k^1 + \mu_a + 1, 0) \;\in\; \mathbb{Q}^2_{\ge 0}\,.$$

The proof in Case (III)(b) is analogous to the proof in (III)(a). As argued above, this completes the proof of Case (III) and also of the entire Proposition. $\square$

Note that Proposition 4.6.3 provides an approach to the proof of Fujita's base point free conjecture for smooth projective irreducible rational varieties with a torus action of complexity one and Picard number two alternative to the one used in Corollary 4.4.14: Instead of using the classification presented in Chapter two, the assertion follows using Proposition 4.6.3 and Remark 4.2.7.

Corollary 4.6.4. *Let $X = X(A, P, u)$ be a non-toric locally factorial projective variety of Picard number two. If $\mathcal{K}_X$ is semiample or if X is log terminal, then X fulfills Fujita's base point free conjecture, Conjecture 4.0.1, i.e. $\mathcal{K}_X + m\mathcal{L}$ is base point free for all $m \ge \dim(X) + 1$ and for all ample Weil divisor classes $\mathcal{L}$.*

Problem 4.6.5. Generalize Proposition 4.6.3 to higher dimensions or find an example of a locally factorial projective variety $X(A, P, u)$ with Picard at least three admitting an ample divisor class $\mathcal{L}$ and an integer $m \ge \dim(X) + 1$ such that $\mathcal{K}_X + (\dim(X) + 1)\mathcal{L}$ is semiample but not base point free.

4.7. Algorithms for embedded monoids

In the following we describe some algorithms for monoids which, applied to Mori dream spaces, can be used for computing generators of the base point free monoid $\mathrm{BPF}(X)$, for testing whether a Weil divisor class is base point free and for computing a point of the conductor ideal of $\mathrm{BPF}(X) \subseteq \mathrm{Pic}(X)$. In [**27**], we provide a maple-based implementation of these algorithms. Note that sections 4.7 – 4.9 have been presented in [**26**].

Algorithm 4.7.1. (inMonoid) *Input:* A finitely generated abelian group K', generators $s_1', \ldots, s_{t'}' \in K'$ of an embedded monoid $S' := \mathrm{lin}_{\mathbb{Z}_{\geq 0}}(s_1', \ldots, s_{t'}') \subseteq K'$ and an element $w' \in K'$.
Output: *True* if w' is contained in S'. Otherwise, *false* is returned.

- By excluding the generators s_i' that equal 0_K, we achieve a representation $S' := \mathrm{lin}_{\mathbb{Z}_{\geq 0}}(s_1', \ldots, s_t')$ with a natural number $t \in \mathbb{Z}_{\leq t'}$ and with non-zero elements s_i'.
- We compute a canonical representation of the embedded monoid $S' \subseteq K'$:
  - Compute $r, \tilde{r} \in \mathbb{Z}_{\geq 0}$ such that there is an isomorphism of groups $\varphi\colon K' \to K := \mathbb{Z}^r \oplus \bigoplus_{k=1}^{\tilde{r}} \mathbb{Z}/a_i\mathbb{Z}$.
  - Let $S := \mathrm{lin}_{\mathbb{Z}_{\geq 0}}(s_1, \ldots, s_t) \subseteq K$, where we set $s_i := \varphi(s_i') \in K$.
  - Set $w := \varphi(w') \in K$.
- Let $Q\colon \mathbb{Z}^t \to K$ denote the homomorphism mapping $x = (x_1, \ldots, x_t) \in \mathbb{Z}^t$ to the integer combination $\sum x_i s_i$. Denote by Q^0 the free part of Q, i.e. with the projection $\pi\colon K \to K^0 = K/K^{\mathrm{tor}}$, we have $\pi \circ Q = Q^0$.
- Compute the polyhedron $\mathcal{B} := (Q^0)^{-1}(w^0) \cap \mathbb{Q}^t_{\geq 0}$.
- If $\mathcal{B}$ is not bounded, then
  - for all $1 \leq i \leq t$ do
    - if $s_i^0 = 0_{K^0}$ holds, then let $\mathcal{C} := \{1 \leq k \leq \tilde{r};\ \ s_{ir+k} \neq 0\}$ and
$$\mathcal{B} \ := \ \mathcal{B} \cap \{x \in \mathbb{Q}^t;\ x_i \leq \prod_{k \in \mathcal{C}} a_k\}\,.$$
- Compute the lattice points of the polytope $\mathcal{B}$, i.e. compute $B := \mathcal{B} \cap \mathbb{Z}^t$.
- Return *true* if there is a point $x \in B$ such that $Q(x) = w$ holds. Otherwise, return *false*.

Proof. We first show that in the end of the above algorithm, the polyhedron $\mathcal{B}$ is a polytope. Note that $s_i \in K$ is a tupel $s_i = (s_{i1}, \ldots, s_{ir}, s_{ir+1}, \ldots, s_{ir+\tilde{r}})$ with integers $s_{ij} \in \mathbb{Z}$, $1 \leq j \leq r$, and elements $s_{ir+k} \in \mathbb{Z}/a_k\mathbb{Z}$, $1 \leq k \leq \tilde{r}$. Via an isomorphism of abelian groups $K \to K$ we may assume that $\mathrm{cone}(S)$ is contained in $\mathbb{Q}^r_{\geq 0}$, i.e. we have $s_{i1}, \ldots, s_{ir} \geq 0$ for all $1 \leq i \leq t$. Consider the polyhedron

$$\mathcal{A} \ := \ (Q^0)^{-1}(w^0) \cap \mathbb{Q}^t_{\geq 0}\,.$$

Note that $\mathcal{A}$ contains exactly those lattice points $x = (x_1, \ldots, x_t) \in \mathbb{Z}^t_{\geq 0}$ with the property that

$$\sum_{i=1}^{t} x_i(s_{i1}, \ldots, s_{ir}) \ = \ \sum_{i=1}^{t} x_i s_i^0 \ = \ Q^0(x) \ = \ w^0 \ = \ (w_1, \ldots, w_r)$$

holds. This means that the integer coefficient x_i is smaller than $\lfloor \frac{w_j}{s_{ij}} \rfloor$ for all $1 \leq j \leq r$ with $s_{ij} \neq 0$, where $\lfloor \cdot \rfloor$ denotes the floor function. In particular, we have

$$\mathcal{A} \ \subseteq \ \left\{ x \in \mathbb{Q}^t_{\geq 0};\ \ x_i \leq \min\left(\lfloor \frac{w_j}{s_{ij}} \rfloor;\ 1 \leq j \leq r,\, s_{ij} \neq 0 \right) \right\}$$

for all $1 \leq i \leq t$ such that $s_i^0 \neq 0_{K^0}$ holds, i.e. $\mathcal{A}$ is bounded with respect to these coordinate directions i. For all other coordinate directions $1 \leq i \leq t$ of $\mathbb{Z}^t$, i.e. of those with $s_i^0 = 0_{K^0}$, the above algorithm computes a bound b_i, where

$$b_i \;:=\; \prod_{k \in \mathcal{C}} a_k \;\in\; \mathbb{Z}, \qquad \mathcal{C} := \{1 \le k \le \tilde{r};\;\; s_{ir+k} \ne 0_{\mathbb{Z}/a_k\mathbb{Z}}\}\,.$$

Note that $\mathcal{C}$ is non-empty since in the first step of the algorithm, we excluded the s_i' that are zero. We conclude that

$$\mathcal{B} \;=\; \mathcal{A} \cap \{x \in \mathbb{Q}^t;\; x_i \le b_i \text{ for all } 1 \le i \le t \text{ with } s_i^0 = 0_{K^0}\}$$

is indeed a polytope and thus $B = \mathcal{B} \cap \mathbb{Z}^t$ is a finite set.

We now explain why the above algorithm has the claimed output. We need to show that $w' \in S'$ holds if and only if the algorithm returns true. Clearly, $w' \in S'$ holds if and only if w is contained in S. This in turn is the case if and only if there is an element $x \in \mathcal{A} \cap \mathbb{Z}^t_{\ge 0}$ such that $Q(x) = w$ holds. If $\mathcal{A}$ is a polytope, there is nothing to show. If $\mathcal{A}$ is unbounded we showed above that there is an index $1 \le i \le t$ such that $s_i^0 = 0_{K^0}$ holds. It remains to show that the following assertions are equivalent:

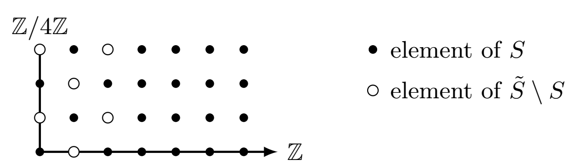
(i) There is an element $x \in \mathcal{A} \cap \mathbb{Z}^t_{\ge 0}$ such that $Q(x) = w$ holds.

(ii) There is an element $y \in B_i := \mathcal{A} \cap \{x \in \mathbb{Z}^t_{\ge 0};\; x_i \le b_i\}$ with $Q(y) = w$.

Since $B_i \subseteq \mathcal{A} \cap \mathbb{Z}^t_{\ge 0}$ holds, the direction "(ii)$\Rightarrow$(i)" is obvious. For the other direction, recall that b_i is the product of all a_k, $1 \le k \le \tilde{r}$, with $s_{ir+k} \ne 0_{\mathbb{Z}/a_k\mathbb{Z}}$. Since $s_i^0 = 0_{K^0}$ holds, we thus obtain $\alpha s_i = \alpha' s_i$ for all integers α, α' with $\alpha \equiv \alpha' (\mathrm{mod}\ b_i)$. This means that it is sufficient to look at coefficient vectors $x \in \mathbb{Z}^t_{\ge 0}$ with $x_i \le b_i$, i.e. (i) implies (ii). As argued above, this completes the proof. $\square$

Example 4.7.2. Consider the abelian group $K := \mathbb{Z} \otimes \mathbb{Z}/4\mathbb{Z}$, its elements $s_1 := (0, \bar{2})$, $s_2 := (1, \bar{1})$, $s_3 := (3, \bar{2})$, $w := (3, \bar{1})$ and the monoid $S := \mathrm{lin}_{\mathbb{Z}_{\ge 0}}(s_1, s_2, s_3)$ depicted in the picture below. Algorithm 4.7.1 applied to S and to w does the following:

- The map Q is defined by $\mathbb{Z}^3 \to K$, $(x_1, x_2, x_3) \mapsto (x_2 + 3x_3, \alpha)$, where we set $\alpha := ((2x_1 + x_2 + 2x_3) + 4\mathbb{Z}) \in \mathbb{Z}/4\mathbb{Z}$. Its free part Q^0 is given by $\mathbb{Z}^3 \to \mathbb{Z}$, $(x_1, x_2, x_3) \mapsto x_2 + 3x_3$.
- The polyhedron $(\mathbb{Q}^0)^{-1}(w^0)$ is given by $\mathbb{Q} \times \{(3 - 3\beta,\, \beta)\,;\; \beta \in \mathbb{Q}\}$. Thus the algorithm starts with the polyhedron
$$\mathcal{B} \;=\; \mathbb{Q}_{\ge 0} \times \{(3 - 3\beta,\, \beta)\,;\; \beta \in \mathbb{Q},\, 0 \le \beta \le 1\}.$$
- Since $\mathcal{B}$ is unbounded and s_i^0 is zero if and only $i = 1$ holds, the algorithm then computes the polytope
$$\mathcal{B} \;:=\; \mathcal{B} \cap \{x \in \mathbb{Q}^t;\; x_1 \le 4\}.$$
Now we have $\mathcal{B} = \{(\alpha,\, 3 - 3\beta,\, \beta)\,;\; \alpha, \beta \in \mathbb{Q},\, 0 \le \alpha \le 4,\, 0 \le \beta \le 1\}$.
- In a next step, the algorithm computes the lattice points B of $\mathcal{B}$:
$$B \;=\; \{(\alpha, 3, 0),\, (\alpha, 0, 1);\; \alpha \in \mathbb{Z},\, 0 \le \alpha \le 4\}.$$
- Since $Q((1, 3, 0)) = 1s_1 + 3s_2 + 0s_3 = w$ holds, the algorithm returns *true*.



Algorithm 4.7.3. (generatorsIntMonoid) *Input:* Two subgroups K_1, K_2 of a finitely generated abelian group K and generators $s_{i1}, \ldots, s_{in_i} \in K_i$ of embedded monoids $S_i := \mathrm{lin}_{\mathbb{Z}_{\ge 0}}(s_{i1}, \ldots, s_{in_i}) \subseteq K_i$, $i = 1, 2$.
Output: A set of generators for the embedded monoid $S_1 \cap S_2 \subseteq K_1 \cap K_2$.

- Let $\varphi := \varphi_1 \times \varphi_2 \colon \mathbb{Z}^{n_1+n_2} \to K \times K$ be the homomorphism of abelian groups defined through $\varphi_i \colon \mathbb{Z}^{n_i} \to K, e_{ij} \mapsto s_{ij}$, where the e_{ij} denote the canonical base vectors of $\mathbb{Z}^{n_i}$. Furthermore, define the projection $\psi \colon K \times K \to (K \times K)/\Delta$, where $\Delta := \{(k,k);\ k \in K\}$ denotes the diagonal.
- Compute the kernel of $\beta := \psi \circ \varphi$.
- Consider the isomorphism of abelian groups $\iota \colon \mathbb{Z}^r \to \ker(\beta)$ and compute generators $g_1, \dots, g_t$ for $\mathbb{Z}^r \cap \iota^{-1}(\mathbb{Q}^{n_1+n_2}_{\geq 0})$.
- Define the projection $\pi \colon K \times K \to K$, $(x,y) \mapsto x$ on the first factor and return the set $\{(\pi \circ \varphi \circ \iota)(g_j);\ j = 1, \dots, t\}$.

Proof. According to Gordan's lemma [**21**, Prop. 1.2.17], there are generators $g_1, \dots g_t$ for the monoid $\mathbb{Z}^r \cap \iota^{-1}(\mathbb{Q}^{n_1+n_2}_{\geq 0})$. Set $M := \ker(\beta) \cap \mathbb{Z}^{n_1+n_2}_{\geq 0}$ and consider the diagram

$$
\begin{tikzcd}
\mathbb{Z}^r \cap \iota^{-1}(\mathbb{Q}^{n_1+n_2}_{\geq 0}) \arrow[r] \arrow[d, phantom, "\cap" rotate=-90] & M \arrow[d, phantom, "\cap" rotate=-90] \arrow[r, phantom, "\subseteq"] & \mathbb{Z}^{n_1+n_2}_{\geq 0} \arrow[r] \arrow[d, phantom, "\cap" rotate=-90] & S_1 \times S_2 \arrow[r] \arrow[d, phantom, "\cap" rotate=-90] & (K \times K)/\Delta \arrow[d, equal] \\
\mathbb{Z}^r \arrow[r, "\iota", "\cong"'] & \ker(\beta) \arrow[r, phantom, "\subseteq"] & \mathbb{Z}^{n_1+n_2} \arrow[r, "\varphi"] \arrow[rr, bend right, "\beta"'] & K \times K \arrow[r, "\psi"] & (K \times K)/\Delta\,.
\end{tikzcd}
$$

With the projection $\pi \colon K \times K \to K$, $(x,y) \mapsto x$ on the first factor, we obtain

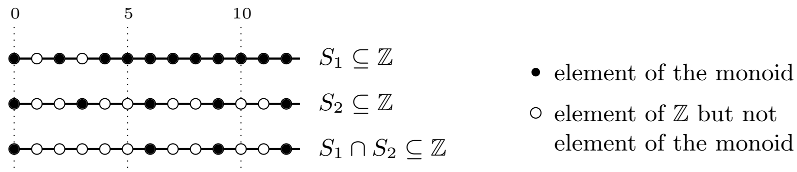
$$
(\pi \circ \varphi \circ \iota)\Big(\mathbb{Z}^r \cap \iota^{-1}(\mathbb{Q}^{n_1+n_2}_{\geq 0})\Big) \;=\; (\pi \circ \varphi)(M) \;=\; S_1 \cap S_2\,,
$$

where the last equality is true since $\varphi(M) = \{(a,b) \in S_1 \times S_2;\ a = b\}$ holds. We conclude that $\{(\pi \circ \varphi \circ \iota)(g_j);\ j = 1, \dots, t\}$ is a set of generators for $S_1 \cap S_2$. $\square$

Example 4.7.4. Consider the abelian group $K_1 := K_2 := K := \mathbb{Z}$ as well as its elements $s_{11} := 2$, $s_{12} := 5$, and $s_{21} := 3$. Algorithm 4.7.3 applied to the monoids $S_1 := \operatorname{lin}_{\mathbb{Z}_{\geq 0}}(s_{11}, s_{12})$ and $S_2 := \operatorname{lin}_{\mathbb{Z}_{\geq 0}}(s_{21})$ depicted in the figure below proceeds as follows:

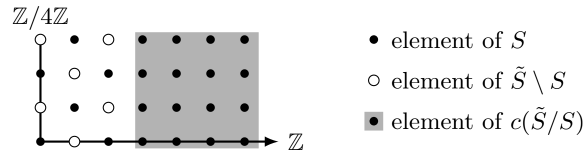
- The map φ is given by $\mathbb{Z}^3 \to \mathbb{Z} \times \mathbb{Z}$, $e_{11} \mapsto s_{11}$, $e_{12} \mapsto s_{12}$, $e_{21} \mapsto s_{21}$, where $3 = 2+1 = n_1+n_2$ holds. To be precise, φ is defined by the matrix
$$
\begin{pmatrix} 2 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}.
$$
- The kernel of β is given by $\ker(\beta) = \operatorname{lin}_{\mathbb{Z}}\big((1,2,4),(0,3,5)\big) \cong \mathbb{Z}^2$.
- The isomorphism $\iota \colon \mathbb{Z}^2 \to \ker(\beta)$ is defined by mapping the first canonical base vector of $\mathbb{Z}^2$ to $(1,2,4)$ and the second one to $(0,3,5)$.
- We have $\mathbb{Q}^2 \cap \iota^{-1}(\mathbb{Q}^3_{\geq 0}) = \operatorname{cone}((3,-2),(0,1))$. According to Gordan's Lemma, computing the lattice points of the polytope
$$
\operatorname{conv}((0,0),(3,-2),(0,1),(3,-1))
$$
gives the following generators for the monoid $\mathbb{Z}^2 \cap \iota^{-1}(\mathbb{Q}^3_{\geq 0})$:
$$
(0,0),\ (0,1),\ (1,0),\ (2,-1),\ (3,-2),\ (3,-1)\,.
$$
- Applying $\pi \circ \varphi \circ \iota$ to those generators gives the generators $0, 15, 12, 9, 6, 21$ for $S_1 \cap S_2$. Note that this list is not a Hilbert basis. To speed up the computation process in [**27**], some reduction mechanisms were implemented.

Algorithm 4.7.5. (inCondIdeal) *Input:* A finitely generated abelian group K, generators $s_1, \dots, s_t \in K$ of an embedded monoid $S := \operatorname{lin}_{\mathbb{Z}_{\geq 0}}(s_1, \dots, s_t) \subseteq K$ and an element $w \in K$.
Output: *True* if w is contained in $c(\tilde{S}/S)$. Otherwise, *false* is returned.



- Compute M as defined in Lemma 4.1.8.
- Use Algorithm 4.7.1 to test whether S contains $w + M$. Return *true* if this is the case; otherwise return *false*.

Proof. Let $w \in K$ and consider M as defined in Lemma 4.1.8. According to this lemma, M generates $\tilde{S}$ as an S-module. This means that the conductor ideal $c(\tilde{S}/S)$ contains w if and only if $w + M$ is contained in S. □



Example 4.7.6. Consider the abelian group $K := \mathbb{Z} \otimes \mathbb{Z}/4\mathbb{Z}$ as well as its elements $s_1 := (0, \bar{2})$, $s_2 := (1, \bar{1})$, $s_3 := (3, \bar{2})$ and the monoid $S := \mathrm{lin}_{\mathbb{Z}_{\geq 0}}(s_1, s_2, s_3)$ as in Example 4.7.2. The monoid and its conductor ideal are illustrated in the above picture. We apply algorithm 4.7.5 to $w := (3, \bar{1})$ and test whether w is contained in $c(\tilde{S}/S)$.

- The maps Q and Q^0 are as in Example 4.7.2.
- The algorithm computes M as defined in Lemma 4.1.8. We obtain

$$M \;=\; \{(0, a),\, (1, a);\; a \in \mathbb{Z}/4\mathbb{Z}\} \;\subseteq\; K\,.$$

- In the next step the algorithm uses Algorithm 4.7.1 to test whether S contains $w + M = \{(3, a),\, (4, a);\; a \in \mathbb{Z}/4\mathbb{Z}\}$.
- Similarily as in Example 4.7.2, for $x \in w + M$ with $x^0 = 3$, Algorithm 4.7.1 computes $B_3 := \{(\alpha, 3, 0),\, (\alpha, 0, 1);\; 0 \leq \alpha \leq 4,\, \alpha \in \mathbb{Z}\}$ and we obtain $B_4 := \{(\alpha, 4, 0),\, (\alpha, 1, 1);\; 0 \leq \alpha \leq 4,\, \alpha \in \mathbb{Z}\}$ for all $x \in w + M$ with $x^0 = 4$. Since for all $x \in w + M$ with $x^0 = i$, $i = 3, 4$, there is some $y \in B_i$ with $Q(y_i) = x_i$, the algorithm returns *true*.

Algorithm 4.7.7. (pointCondIdeal) *Input:* A finitely generated abelian group K, an element $w \in K$ and generators $s_1, \ldots, s_t \in K$ of a spanning embedded monoid $S := \mathrm{lin}_{\mathbb{Z}_{\geq 0}}(s_1, \ldots, s_t) \subseteq K$.
Output: A point of the conductor ideal $c(\tilde{S}/S)$.

- Compute $w \in K$ that defines a point in the relative interior of $\mathrm{cone}(S)$.
- Use Algorithm 4.7.5 to compute the smallest integer $r \in \mathbb{Z}_{\geq 1}$ such that rw is contained in $c(\tilde{S}/S)$. Return rw.

Proof. This Algorithm terminates since $S \subseteq K$ is spanning. □

Example 4.7.8. Consider the abelian group $K := \mathbb{Z} \otimes \mathbb{Z}/4\mathbb{Z}$ as well as its elements $s_1 := (0, \bar{2})$, $s_2 := (1, \bar{1})$, $s_3 := (3, \bar{2})$ and the monoid $S := \mathrm{lin}_{\mathbb{Z}_{\geq 0}}(s_1, s_2, s_3)$ as in Examples 4.7.2 and 4.7.6. We apply algorithm 4.7.7 to compute an element of $c(\tilde{S}/S)$.

- At first the algorithm computes the element $(1, \bar{0}) \in K$ defining an element in the relative interior of $\mathrm{cone}(S)$.

- For $j = 1, 2$, Algorithm 4.7.5 returns that $j\,(1, \bar{0})$ is not contained in the conductor ideal $c(\tilde{S}/S)$.
- In the next step, Algorithm 4.7.5 shows that $(3, \bar{0})$ is an element of $c(\tilde{S}/S)$.

Here comes an example computation.

Example 4.7.9. We consider the embedded monoid $S \subseteq K := \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ generated by $[3, \bar{0}], [5, \bar{0}], [5, \bar{1}], [3, \bar{2}] \in K$ and perform some monoid membership and conductor ideal membership tests. Furthermore, we compute an element of the conductor ideal of $S \subseteq K$.

```
> S := matrix([[3,5,5,3],[0,0,1,2]]);
```

$$S := \begin{bmatrix} 3 & 5 & 5 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

```
> K := createAG(1,[3]);
```

$$K := AG(1, [3])$$

```
> inMonoid(S,[6,2],K); inMonoid(S,[7,0],K); inMonoid(S,[8,2],K);
```

$$true$$
$$false$$
$$true$$

```
> inCondIdeal(S,[6,2],K); inCondIdeal(S,[7,0],K); inCondIdeal(S,[8,2],K);
```

$$false$$
$$false$$
$$true$$

```
> pointCondIdeal(S,K);
```

$$[8, 0]$$

We now compute generators for the intersection of the monoids $S_1, S_2 \subseteq K$ generated by $[3, \bar{0}], [5, \bar{0}], [5, \bar{1}] \in K$ and $[3, \bar{0}], [5, \bar{0}], [3, \bar{2}] \in K$, respectively.

```
> generatorsIntMonoid(S,[{1,2,3},{1,2,4}],K);
```

$$[[3, 0], [5, 0], [11, 1], [13, 2], [15, 1], [15, 2]]$$

4.8. Algorithms for the base point free monoid of Mori dream spaces

Here we apply the algorithms of the previous section for computing generators of the base point free monoid and for testing whether a Weil divisor class is base point free or not. The implementation of the following algorithms builds on the maple-based software package `MDSpackage` **[38]**. A Mori dream space X is entered and stored in terms of an ample class u together with pairwise non-associated $\mathrm{Cl}(X)$-prime generators and the relations of $\mathrm{Cox}(X)$. As explained above, this data fixes a Mori dream space up to isomorphism.

Algorithm 4.8.1. (generatorsBPF) *Input:* A Mori dream space $X(R, \mathfrak{F}, \Phi)$.
Output: A set of generators for the embedded monoid $\mathrm{BPF}(X) \subseteq \mathrm{Pic}(X)$.

- Use `MDSpackage` to compute the covering collection of X.
- Use Algorithm 4.7.3 to compute generators of the intersection

$$\bigcap_{\gamma_0 \in \mathrm{cov}(\Phi)} Q(\gamma_0 \cap E)\,.$$

Algorithm 4.8.2. (isBasePointFree) *Input:* A Mori dream space X and a Weil divisor class $w \in \mathrm{Cl}(X)$.
Output: *True* if w is base point free. Otherwise, *false* is returned.

- Use Algorithm 4.8.1 to compute generators of $\mathrm{BPF}(X) \subseteq \mathrm{Pic}(X)$.
- Apply Algorithm 4.7.1 to w and $\mathrm{BPF}(X)$.

Algorithm 4.8.3. (BPFisSaturated) *Input:* A locally factorial Mori dream space X.

- Use `convex` **[29]** to compute a Hilbert basis $u_1, \ldots, u_t$ of the semiample cone $\mathrm{SAmple}(X) \subseteq \mathrm{Pic}(X) = \mathrm{Cl}(X)$.

- Use Algorithm 4.8.2 to test whether $u_1, \ldots, u_t$ are base point free.

Using the implementation given in [**27**], we study the question of the existence of semiample Cartier divisor classes that are not base point free. It is well-known that for Cartier divisors on complete toric varieties, semiampleness implies base point freeness, see for instance [**21**, Theorem 6.3.12.]. For smooth rational projective varieties with a torus action of complexity one and Picard number two, the same statement follows immediately from the classification done in [**28**]. Note that the discrepancy between semiampleness and base point freeness of divisors on varieties with a torus action of complexity one is already fairly well understood in the language of polyhedral divisors: A criterion for semiampleness is given in [**59**, Theorem 3.27] and a criterion for base point freeness was proven in [**41**, Theorem 3.2].

Example 4.8.4. We give an example of a smooth Mori dream $\mathbb{K}^*$-surface that admits semiample Cartier divisor classes with base points.

```
> Q := matrix([[1,-1,-1,0,0,0,0,0,0,0,0,0,0,0,0],[0,1,-1,1,0,0,0,0,0,0,0,0,0,0,0],
  [0,1,0,-1,1,0,0,0,0,0,0,0,0,0,0],[0,1,0,0,-1,1,0,0,0,0,0,0,0,0,0],[0,0,0,0,0,0,
  -1,1,1,0,0,0,0,0,0],[0,-1,0,0,0,1,0,-1,1,0,0,0,0,0,0],[0,0,0,1,0,0,1,0,1,1,0,0,
  0,0,0],[0,1,0,0,0,0,0,0,1,0,1,0,0,0,0],[1,0,0,-1,0,0,1,0,0,0,0,1,0,0,0],[0,1,0,
  0,0,0,0,1,0,0,0,0,1,0,0],[0,1,0,0,0,-1,0,0,0,0,0,0,0,1,0],[0,-1,0,0,0,1,0,0,0,0,
  0,0,0,0,1]]);
```

$$Q := \begin{bmatrix} 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

```
> RL := [T[1]^5*T[2]*T[3]^4*T[4]^3*T[5]^2*T[6]+T[7]^2*T[8]*T[9]
  +T[10]^3*T[11]*T[12]^2*T[13]];
```

$$RL := \left[T_1^5 T_2 T_3^4 T_4^3 T_5^2 T_6 + T_7^2 T_8 T_9 + T_{10}^3 T_{11} T_{12}^2 T_{13}\right]$$

```
> R := createGR(RL, vars(15), [Q]);
```

$$R := GR(15, 1, [12, []])$$

```
> X := createMDS(R,relint(MDSmov(R)));
```

$$X := MDS(15, 1, 2, [12, []])$$

```
> MDSissmooth(X);
```

$$true$$

```
> COV := MDScov(X);
```

$$\begin{aligned} COV := [&\{3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}, \{2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}, \\ &\{1, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}, \{1, 2, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}, \\ &\{1, 2, 3, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}, \{1, 2, 3, 4, 7, 8, 9, 10, 11, 12, 13, 14, 15\}, \\ &\{1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 13, 15\}, \{1, 2, 3, 4, 5, 6, 9, 10, 11, 12, 13, 14, 15\}, \\ &\{1, 2, 3, 4, 5, 6, 8, 10, 11, 12, 13, 14, 15\}, \{1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 15\}, \\ &\{1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14\}, \{1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 13, 14, 15\}, \\ &\{1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 14, 15\}, \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 13, 14, 15\}, \\ &\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 15\}, \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14\}] \end{aligned}$$

```
> w := [-1,1,1,1,3,2,3,4,0,3,1,5];
```

$$w := [-1, 1, 1, 1, 3, 2, 3, 4, 0, 3, 1, 5]$$

```
> contains(MDSsample(X),w);
```

$$true$$

```
> isBasePointFree(X,w);
```

$$false$$

The computation shows that $w = [-1, 1, 1, 1, 3, 2, 3, 4, 0, 3, 1, 5]$ is a semiample but not base point free Cartier divisor class.

For a geometric interpretation note that X is obtained by blowing up $\mathbb{P}_1 \times \mathbb{P}_1$ ten times in the following way: One considers the $\mathbb{K}^*$-action on $\mathbb{P}_1 \times \mathbb{P}_1$ given by

$$t \cdot ([y_0, y_1], [z_0, z_1]) \ := \ ([y_0, y_1], [z_0, tz_1]) .$$

The fixed points lie on the two curves $C_1 := \mathbb{P}_1 \times \{[0,1]\}$ and $C_1 := \mathbb{P}_1 \times \{[1,0]\}$. In order two obtain X, one blows up the three fixed points $c_{11} := [[0,1],[0,1]] \in C_1$, $c_{12} := [[1,0],[0,1]] \in C_1$ and $c_{21} := [[1,-1],[1,0]] \in C_2$. The resulting hyperbolic fixed points are again blown up: for c_{11}, one repeats this procedure four times, for c_{12}, one repeats this procedure two times and for c_{21} just once. The resulting variety then is isomorphic to X.

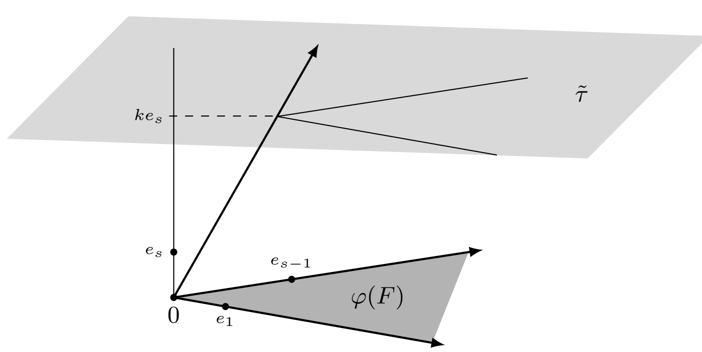
4.9. Fujita base point free test algorithm

In order to test whether a $\mathbb{Q}$-factorial Mori dream space X with known canonical class fulfills Fujita's base point free conjecture, Conjecture 4.0.1, we need to test whether $\mathcal{K}_X + m\mathcal{L}$ is an element of $\mathrm{BPF}(X)$ for all $m \geq \dim(X) + 1$ and for all ample Cartier divisor classes $\mathcal{L}$. Since we can only carry out finitely many tests, we encounter two problems: firstly, we need to bound m and secondly, we need to find a finite validation set of Cartier divisor classes $\mathcal{L}$. In this section, we introduce our solution to these problems and also present some examples of applying our test algorithm.

Remark 4.9.1. Algorithm 4.9.4 applies to Mori dream spaces with known canonical class. For instance, if $\mathrm{Cox}(X)$ is a complete intersection, there is a concrete formula for the canonical class in terms of generators and relations of $\mathrm{Cox}(X)$ [**3**, Prop. 3.3.3.2]. Note that all irreducible normal rational projective varieties with a torus action of complexity one have a complete intersection Cox ring [**37**, Prop. 1.2]. Moreover, there are formulas for the canonical class of spherical varieties [**16, 50**].

Construction 4.9.2. Let K^0 be a lattice. Consider an s-dimensional cone $\sigma \subseteq K^0_{\mathbb{Q}}$ with some facet $F \preceq \sigma$. Let $\varphi\colon K^0 \to \mathbb{Z}^n$ be an isomorphism of $\mathbb{Z}$-modules such that $\varphi(\sigma) \subseteq \mathrm{cone}(e_1, \ldots, e_s)$ and $\varphi(F) \subseteq \mathrm{cone}(e_1, \ldots, e_{s-1})$ holds, where $e_1, \ldots, e_n$ denote the canonical base vectors of the rational vector space $\mathbb{Q}^n$. For any $k \in \mathbb{Z}$ we call $\tau := \varphi^{-1}(\tilde{\tau})$ the *k-th facet parallel* of F, where we set

$$\tilde{\tau} \ := \ \left(\mathrm{lin}_{\mathbb{Q}}(\varphi(F)) \ + \ ke_s\right).$$



Setting 4.9.3. Let X be a $\mathbb{Q}$-factorial Gorenstein Mori dream space and consider the base point free monoid $S := \mathrm{BPF}(X) \subseteq K := \mathrm{Pic}(X)$. We denote by $F_1, \ldots, F_r$ the facets of $\sigma := \mathrm{cone}(w^0 \otimes 1;\ w \in S) \subseteq K^0_{\mathbb{Q}}$. Consider an index $1 \leq i \leq r$ and let $m_1, \ldots, m_{n_i} \in S$ be those elements such that m_j^0 is minimal with the property

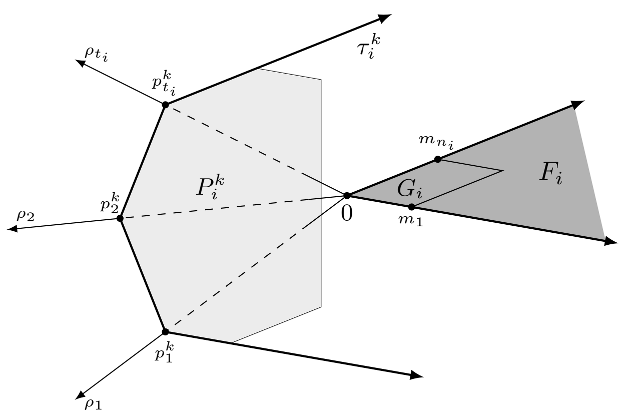
that $m_j^0 \otimes 1$ is contained in a ray of F_i. Consider the polytope

$$G_i \; := \; \{\sum_{j=1}^{n_i} a_j \, (m_j^0 \otimes 1); \;\; a_j \in \mathbb{Q}, \; 0 \leq a_j \leq 1\} \; \subseteq \; F_i$$

as indicated in the figure below and let $\rho_1, \dots \rho_{t_i}$ be the rays of σ that are not contained in F_i. We denote by τ_i^k the k-th facet parallel of F_i. For each facet parallel τ_i^k with $k \in \mathbb{Z}_{\geq 0}$, we denote by $p_j^k \in K_{\mathbb{Q}}$, $1 \leq j \leq t_i$, the point that is the intersection of ρ_j and τ_i^k. With the canonical embedding $\iota_0 \colon K^0 \to K^0_{\mathbb{Q}}$, $w \mapsto w \otimes 1$, we define

$$\begin{aligned} P_i^k \;\; &:= \;\; \big(\mathrm{conv}(p_1^k, \dots, p_{t_i}^k) + G_i\big) \; \cap \; \sigma^\circ \; \subseteq \; \tau_i^k \quad \text{and} \\ Gp_i^k \;\; &:= \;\; \iota_0^{-1}\left(P_i^k\right) \; \times \; K^{\mathrm{tor}} \; \subseteq \; K \end{aligned}$$

for all $k \in \mathbb{Z}_{\geq 0}$, where σ° denotes the relative interior of σ. Consider the canonical class $\mathcal{K}_X \in K$ of X. Since $S \subseteq K$ is spanning, there is an element $C \in c(\tilde{S}/S)$. For $1 \leq i \leq r$ let α_i be an integer such that $(-\mathcal{K}_X^0 + C^0) \otimes 1 \in \tau_i^{\alpha_i}$ holds and set $\nu := \max(\alpha_i;\; 1 \leq i \leq r)$. Note that α_i may be negative.



The above mentioned problems, namely bounding m and finding a finite validation set of Cartier divisor classes, are tackled by computing a point of the conductor ideal of $\mathrm{BPF}(X)$ and by only considering the Cartier divisor classes defining a point in the polytopes P_i^k of the first few facet parallels τ_i^k, $k \geq 0$, of each facet $F_i \preceq \sigma$.

Algorithm 4.9.4. (fujitaBpf) *Input:* A $\mathbb{Q}$-factorial Mori dream space X and its canonical class $\mathcal{K}_X$.
Output: *True* if X fulfills Fujita's base point free conjecture, i.e. if $\mathcal{K}_X + m\mathcal{L}$ is base point free for all $m \geq \dim(X) + 1$ and all ample Cartier divisor classes $\mathcal{L}$. Otherwise, *false* is returned.

- If X is not Gorenstein return *false*.
- Use Algorithm 4.7.3 to compute generators of $S := \mathrm{BPF}(X)$.
- Use Algorithm 4.7.7 to compute a point $C \in c(\tilde{S}/S)$.
- Compute the facets $F_1, \dots, F_r$ of cone(S) and $\alpha_1, \dots, \alpha_r$ as well as ν as defined in Setting 4.9.3.
- For each $1 \leq i \leq r$ do
  - for each $\dim(X) + 1 \leq m \leq \nu - 1$ do
    - for each $1 \leq k \leq \lfloor \frac{\alpha_i - 1}{m} \rfloor$, where $\lfloor \cdot \rfloor$ denotes the floor function, use Algorithm 4.7.1 to test whether $\mathcal{K}_X + m\, Gp_i^k \subseteq S$ holds.
- Return *false* if there is $1 \leq i \leq r$, $\dim(X) + 1 \leq m \leq \nu - 1$, $1 \leq k \leq \lfloor \frac{\alpha_i - 1}{m} \rfloor$, and $\mathcal{L} \in Gp_i^k$ such that $\mathcal{K}_X + m\mathcal{L}$ is not contained in S. Otherwise, return *true*.

Before presenting a proof of Algorithm 4.9.4, we first give two examples of applying it to Mori dream spaces.

Example 4.9.5. Here we give an example of a six-dimensional smooth Mori dream space that does fulfill Fujita's base point free conjecture.

```
> Q := matrix([[1,1,2,0,1,1,1,-1,0,0],[0,0,-1,1,0,-1,-1,1,0,0],[0,36,36,0,18,49,49,
  -48,1,1]]);
```

$$Q = \begin{bmatrix} 1 & 1 & 2 & 0 & 1 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & -1 & -1 & 1 & 0 & 0 \\ 0 & 36 & 36 & 0 & 18 & 49 & 49 & -48 & 1 & 1 \end{bmatrix}$$

```
> RL := [T[1]*T[2]+T[3]*T[4]+T[5]^2];
```

$$RL := \left[T_1 T_2 + T_3 T_4 + T_5^2\right]$$

```
> R := createGR(RL,vars(10),[Q]);
```

$$R := GR(10, 1, [3, []])$$

```
> X := createMDS(R,[1,1,50]);
```

$$X := MDS(10, 1, 6, [3, []])$$

```
> MDSissmooth(X);
```

$$true$$

Since $R = \mathrm{Cox}(X)$ is a complete intersection, we may use the formula presented in **[3]** to compute the canonical class of X: we obtain $\mathcal{K}_X = [-4, 1, -106] \in \mathbb{Z}^3$.

```
> fujitaBPF(X,[-4,1,-106]);
```

$$true$$

To obtain this result the algorithm performs the following steps:

- First Algorithm 4.7.3 is used to compute the three generators of $[0, 0, 1]$, $[0, 1, 0]$ and $[1, 0, 49]$ of $\mathrm{BPF}(X) \subseteq \mathbb{Z}^3$.
- Then Algorithm 4.7.7 computes the point $C := [0, 0, 0] \in c(\tilde{S}/S)$.
- The faces of $\mathrm{cone}(S)$ are given by $F_1 := \mathrm{cone}([0, 1, 0], [1, 0, 49])$, $F_2 := \mathrm{cone}([0, 0, 1], [1, 0, 49])$, $F_3 := \mathrm{cone}([0, 1, 0], [0, 0, 1])$. The algorithms then computes $\alpha_1, \ldots, \alpha_3$ such that $-\mathcal{K}_X + C = [4, -1, 106]$ defines a point in $\tau_i^{\alpha_i}$. We obtain $\alpha_1 = -90$, $\alpha_2 = -1$ and $\alpha_3 = 4$ as well as $\nu = 4$. Note that $\alpha_3 = 4$ is just the first coordinate of $-\mathcal{K}_X + C$.
- Since $\dim(X) + 1 = 6 > 4 = \nu - 1$ holds, the algorithm returns *true*.

For a geometric description of X, note that X admits three elementary contractions two of which are birational small. The other one is a birational divisorial contraction $X \to Y$ contracting the divisor corresponding to the variable T_8 of $\mathrm{Cox}(X)$. The variety Y is a smooth intrinsic quadric with generator degrees, relation and semiample cone given by

$$Q \;=\; \left[\begin{array}{cc|cc|c||cccc} 60 & 0 & 48 & 12 & 30 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{array}\right], \qquad g \;=\; T_1T_2 + T_3T_4 + T_5^2$$

and $\mathrm{SAmple}(X) = \mathrm{cone}((1, 0), (60, 1))$. The center of φ is the intersection of Y and the toric prime divisors corresponding to the variables $T_8, T_9 \in \mathrm{Cox}(Y)$. Note that Y allows a closed embedding into the projectivized split vector bundle

$$\mathbb{P}\big(\mathcal{O}_{\mathbb{P}_3} \oplus \mathcal{O}_{\mathbb{P}_3}(12) \oplus \mathcal{O}_{\mathbb{P}_3}(30) \oplus \mathcal{O}_{\mathbb{P}_3}(48) \oplus \mathcal{O}_{\mathbb{P}_3}(60)\big)\,.$$

Example 4.9.6. Here we give an example of a locally factorial variety with a torus action of complexity one that does not fulfill Fujita's base point free conjecture. Note that this represents a difference to the toric case, where Fujino **[30]** presented a proof of Fujita's base point free conjecture for toric varieties with arbitrary singularities.

```
> Q := matrix([[0,0,1,0,0,1,1,0,1],[1,1,0,1,1,0,1,1,2]]);
```

$$Q = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 2 \end{bmatrix}$$

```
> RL := [T[1]*T[2]^7*T[3]^8 +T[4]*T[5]^7*T[6]^8+T[7]^8];
```

$$RL := \left[T_1 T_2^7 T_3^8 + T_4 T_5^7 T_6^8 + T_7^8\right]$$

```
> R := createGR(RL,vars(9),[Q]);
```

$$R := GR(9, 1, [2, []])$$

```
> X := createMDS(R,[1,3]);
```

$$X := MDS(9, 1, 6, [2, []])$$

```
> MDSisfact(X);
```

$$true$$

```
> MDSisquasismooth(X);
```

$$false$$

Since $\mathrm{Cox}(X)$ is a complete intersection, we may use the formula presented in **[3]** to compute the canonical class of X: we obtain $\mathcal{K}_X = [4, 0] \in \mathbb{Z}^2$.

```
> fujitaBPF(X,[4,0]);
```

$$false$$

```
> isBasePointFree(X,[1,3]);
```

$$true$$

Note that Algorithm 4.9.4 returns false, i.e. X does not fulfill Fujita's base point free conjecture. To obtain this result the algorithm performs the following steps:

- First Algorithm 4.7.3 is used to compute the generators $[0, 1]$ and $[1, 2]$ of $\mathrm{BPF}(X) \subseteq \mathbb{Z}^3$.
- Then Algorithm 4.7.7 computes the point $C := [0, 0] \in c(\tilde{S}/S)$.
- The faces of $\mathrm{cone}(S)$ are given by $F_1 := \mathrm{cone}([1, 2])$, $F_2 := \mathrm{cone}([0, 1])$. The algorithms then computes α_1, α_2 such that $-\mathcal{K}_X + C = [-4, 0]$ defines a point in $\tau_i^{\alpha_i}$. We obtain $\alpha_1 = 8$, $\alpha_2 = -4$ and $\nu = 8$. Note that $\alpha_2 = -4$ is just the first coordinate of $-\mathcal{K}_X + C$.
- Then the algorithm performs the following steps:
  - Since we have $\dim(X) + 1 = 7 \leq m \leq 7 = \nu - 1$, the algorithm only needs to test the case $m = 7$.
    - For $i = 1$ we have $\lfloor \frac{\alpha_1 - 1}{7} \rfloor = 1$, i.e. only the case $k = 1$ needs to be considered. The algorithm yields $Gp_i^k = \{[1, 3]\}$.
    - Now Algorithm 4.7.1 is used to test whether $\mathcal{K}_X + m\, Gp_i^k \subseteq S$ holds. We have $\mathcal{K}_X + 7\,[1, 3] = [11, 21]$ which is not contained in $\mathrm{cone}(S)$. Thus Algorithm 4.7.1 returns false.
- Hence the algorithm fujitaBPF returns *false*.

Note that the $\mathcal{K}_X + 7\,[1, 3] = [11, 21]$ is not semiample and thus not nef. Maeda proved in **[51**, Proposition 2.1**]** that $\mathcal{K}_X + m\mathcal{L}$ is nef for all $m \geq \dim(X) + 1$ and for all $\mathcal{L} \in \mathrm{Ample}(X) \cap \mathrm{Pic}(X)$ if X is an irreducible normal projective variety with at most log terminal singularities. Nevertheless, this example does not contradict the result of Maeda since X is not log terminal: To see this, one can look at the affine variety $X_{\gamma_{14}}$. By **[2]**, $X_{\gamma_{14}}$ is log terminal only if the exponents of different monomials are platonic triples. Since this is not the case, we conclude that X is not log terminal.

Observe that the base point free monoid $\mathrm{BPF}(X) \subseteq \mathbb{Z}^2$ is saturated and thus the ample class $[1, 3]$ is base point free. Although $\mathcal{K}_X + 7\,[1, 3] = [11, 21]$ is not base point free on X, a result of **[44]** implies that $\mathcal{K}_X + 7\,[1, 3] = [11, 21]$ is very ample and thus base point free on X^{reg}.

For a geometric description of X, note that X admits an elementary contraction $\varphi\colon X \to \mathbb{P}_4$ of fiber type with fibers isomorphic to a hypersurface of degree eight in $\mathbb{P}_3$. To be precise we have $\varphi^{-1}(a) \cong V_{\mathbb{P}_3}(a_1 a_2^7 T_0^8 + a_3 a_4^7 T_1^8 + T_2^8)$ where $a = [a_1, \ldots, a_5] \in \mathbb{P}_4$ denotes a point of $\mathbb{P}_4$ in homogeneous coordinates and where

T_0, T_1, T_2, T_3 denote the coordinates of $\mathrm{Cox}(\mathbb{P}_3)$. Moreover, X admits a closed embedding $X \to Y$ into the projectivized split vector bundle

$$Y \;=\; \mathbb{P}\big(\mathcal{O}_{\mathbb{P}_4} \oplus \mathcal{O}_{\mathbb{P}_4} \oplus \mathcal{O}_{\mathbb{P}_4}(1) \oplus \mathcal{O}_{\mathbb{P}_4}(2)\big)\,.$$

We now turn to the proof of Algorithm 4.9.4.

Lemma 4.9.7. *In the setting of 4.9.3, the following are equivalent:*

(i) $\mathcal{K}_X + m\mathcal{L} \in S$ *holds for all* $m \geq \dim(X) + 1$ *and for all ample Cartier divisor classes* $\mathcal{L}$*, i.e.* X *fulfills Fujita's base point free conjecture.*
(ii) $\mathcal{K}_X + m\mathcal{L} \in S$ *holds for all* $\nu - 1 \geq m \geq \dim(X) + 1$ *and for all ample Cartier divisor classes* $\mathcal{L}$*.*

Proof. Only implication "(ii)⇒(i)" needs to be proven. Consider $m \geq \dim(X) + 1$. If $m \leq \nu - 1$ holds, then $\mathcal{K}_X + m\mathcal{L} \in S$ follows by (ii). Now assume that $m \geq \nu$ holds. Note that since $\mathcal{L}$ defines a point in the relative interior of σ for all $1 \leq i \leq r$, the multiple $m\mathcal{L}^0 \otimes 1$ is contained in a facet parallel $\tau_i^{\beta_i}$ with $\beta_i \geq m \geq \nu$. Thus by definition of ν as maximum over all integers α_i with $(-\mathcal{K}_X^0 + C^0) \otimes 1 \in \tau_i^{\alpha_i}$, we obtain

$$m\mathcal{L} \otimes 1 \;\in\; ((-\mathcal{K}_X + C) \otimes 1) \;+\; \mathrm{cone}(S)\,.$$

Thus, $\mathcal{K}_X + m\mathcal{L}$ defines a point in $(C \otimes 1) + \mathrm{cone}(S)$. Since C is an element of the conductor ideal $c(\tilde{S}/S)$ of $S \subseteq K$, we conclude $\mathcal{K}_X + m\mathcal{L} \in S$. □

Lemma 4.9.8. *In the setting of 4.9.3, the following are equivalent for* $m \in \{\dim(X)+1, \ldots, \nu - 1\}$:

(i) $\mathcal{K}_X + m\mathcal{L} \in S$ *holds for all ample Cartier divisor classes* $\mathcal{L}$*.*
(ii) *For all* $1 \leq i \leq r$ *and for all* $1 \leq k \leq \lfloor \frac{\alpha_i - 1}{m} \rfloor$*, where* $\lfloor \cdot \rfloor$ *denotes the floor function, we have* $\mathcal{K}_X + m\mathcal{L} \in S$ *for all* $\mathcal{L} \in \iota_0^{-1}(\tau_i^k \cap \sigma^\circ) \times K^{\mathrm{tor}}$*.*

Proof. Only implication "(ii)⇒(i)" needs to be proven. Consider an ample Cartier divisor class $\mathcal{L}$, i.e.

$$\mathcal{L} \;\in\; \iota_0^{-1}(\sigma^\circ) \times K^{\mathrm{tor}}$$

holds. Denote by $\beta_1, \ldots, \beta_r \in \mathbb{Z}_{>0}$ positive integers such that $\mathcal{L}^0 \otimes 1 \in \tau_i^{\beta_i}$ holds. If $\beta_i \leq \lfloor \frac{\alpha_i - 1}{m} \rfloor$ holds for some $1 \leq i \leq r$, then $\mathcal{K}_X + m\mathcal{L} \in S$ follows by (ii). Now assume that $\beta_i > \lfloor \frac{\alpha_i - 1}{m} \rfloor$ holds for all $1 \leq i \leq r$. We obtain $m\beta_i \geq \alpha_i$ for all $1 \leq i \leq r$. Recall that $(-\mathcal{K}_X^0 + C^0) \otimes 1 \in \tau_i^{\alpha_i}$ holds for all $1 \leq i \leq r$. Thus $m\beta_i \geq \alpha_i$ for all $1 \leq i \leq r$ shows that

$$m\mathcal{L} \otimes 1 \;\in\; ((-\mathcal{K}_X + C) \otimes 1) \;+\; \mathrm{cone}(S)$$

holds. Thus, $\mathcal{K}_X + m\mathcal{L}$ defines a point in $(C \otimes 1) + \mathrm{cone}(S)$. Since C is an element of the conductor ideal $c(\tilde{S}/S)$ of $S \subseteq K$, we conclude $\mathcal{K}_X + m\mathcal{L} \in S$. □

Lemma 4.9.9. *In the setting of 4.9.3, consider indices* $1 \leq i \leq r$, $1 \leq k \leq \lfloor \frac{\alpha_i - 1}{m} \rfloor$ *and an ample Cartier divisor class* $\mathcal{L} \in \iota_0^{-1}(\tau_i^k \cap \sigma^\circ) \times K^{\mathrm{tor}}$*. Then there are* $y \in Gp_i^k$ *and* $a_j \in \mathbb{Z}_{\geq 0}$ *such that we have*

$$\mathcal{L} \;=\; y + \sum_{j=1}^{n_i} a_j m_j\,.$$

Proof. Observe that $\sigma \cap \tau_i^k = \mathrm{conv}(p_1^k, \ldots, p_{t_i}^k) + \mathrm{cone}(G_i)$ holds. Hence there are rational numbers a_j, $b_\ell \in \mathbb{Q}_{\geq 0}$, $\sum_{j=1}^{t_i} a_j = 1$, such that

$$\mathcal{L} \;=\; \left(\sum_{j=1}^{t_i} a_j p_j^k + \sum_{\ell=1}^{n_i} b_\ell m_\ell^0,\;\; \mathcal{L}^{\mathrm{tor}}\right)$$

holds. We obtain $\mathcal{L} = y + \sum_{\ell=1}^{n_i} \lfloor b_\ell \rfloor m_\ell$ (4.9.9.1), where $\lfloor \cdot \rfloor$ denotes the floor function and where y is given as

$$y \;:=\; \Big(\sum_{j=1}^{t_i} a_j p_j^k,\;\; \mathcal{L}^{\mathrm{tor}} - \sum_{\ell=1}^{n_i} b_\ell m_\ell^{\mathrm{tor}}\Big) \;+\; \sum_{\ell=1}^{n_i} (b_\ell - \lfloor b_\ell \rfloor)\, m_\ell\,.$$

Note that y is an element of K since we have $y = \mathcal{L} - \sum_{\ell=1}^{n_i} \lfloor b_\ell \rfloor m_\ell$, where $\mathcal{L}$ as well as the m_ℓ, $1 \le \ell \le n_i$, are elements of K. If $y^0 \otimes 1 \in \sigma^\circ$ holds, (4.9.9.1) is the required representation of $\mathcal{L}$. Now consider the case where $y^0 \otimes 1$ is not contained in σ°. This means that $y^0 \otimes 1 \in \big(\mathrm{conv}(p_1^k, \ldots, p_{t_i}^k) \setminus \sigma^\circ\big)$ holds. Since $\mathcal{L}^0 \otimes 1$ is contained in σ°, there is $1 \le \ell \le n_i$ with $\lfloor b_\ell \rfloor \neq 0$. Without loss of generality we assume that $\lfloor b_1 \rfloor, \ldots, \lfloor b_{\ell_0} \rfloor > 0$ and $\lfloor b_{\ell_0+1} \rfloor = \ldots = \lfloor b_{\ell_{n_i}} \rfloor = 0$ hold for some $1 \le \ell_0 \le n_i$. Then we have

$$\mathcal{L} = y' + \sum_{j=1}^{\ell_0} (\lfloor b_j \rfloor - 1)\, m_j \;\; (4.9.9.2)\,, \qquad \text{where} \qquad y' \;:=\; y + \sum_{j=1}^{\ell_0} m_j$$

holds. In order to show that formula (4.9.9.2) is the required representation of $\mathcal{L}$, it remains to prove that $y' \in G_{p_i}^k$ holds. Note that $y' \in K$ holds since y is an element of K. Moreover, since $y^0 \otimes 1 \in \big(\mathrm{conv}(p_1^k, \ldots, p_{t_i}^k) \setminus \sigma^\circ\big)$ holds, y' defines a point in $\mathrm{conv}(p_1^k, \ldots, p_{t_i}^k) + G_i$. It remains to show that y' defines a point in the relative interior of σ. Recall that $\sum_{j=1}^{\ell} m_j^0 \otimes 1$ is contained in the facet F_i. Furthermore, since we are in the case $y^0 \otimes 1 \notin \sigma^\circ$, the point $y^0 \otimes 1$ lies in a facet F_y of σ. Since $k \ge 1$ and $y \in \iota_0^{-1}(\tau_i^k)$ hold, we conclude that $y^0 \otimes 1$ is not contained in F_i, i.e. there is no face $\kappa \preceq \sigma$ with $y^0 \otimes 1 \in \kappa$ and $\sum_{j=1}^{\ell} m_j^0 \otimes 1 \in \kappa$. Thus the sum $y^0 + \sum_{j=1}^{\ell} m_j^0$ defines a point in the relative interior of σ. As argued above, this shows that y' is an element of $G_{p_i}^k$, which completes the proof. □

Lemma 4.9.10. *In the setting of 4.9.3, consider* $\dim(X) + 1 \le m \le \nu - 1$, $1 \le i \le r$ *and* $1 \le k \le \lfloor \frac{\alpha_i - 1}{m} \rfloor$. *Then the following are equivalent:*

(i) $\mathcal{K}_X + m\mathcal{L} \in S$ *holds for all* $\mathcal{L} \in \iota_0^{-1}\left(\tau_i^k \cap \sigma^\circ\right) \times K^{\mathrm{tor}}$.
(ii) $\mathcal{K}_X + m\mathcal{L} \in S$ *holds for all* $\mathcal{L} \in Gp_i^k$.

Proof. Since $Gp_i^k \subseteq \iota_0^{-1}\left(\tau_i^k \cap \sigma^\circ\right) \times K^{\mathrm{tor}}$ holds, only implication "(ii)⇒(i)" needs to be proven. Note that this is an immediate consequence of Lemma 4.9.9. □

Proof of Algorithm 4.9.4. We need to show that X fulfills Fujita's base point free conjecture if and only if the above algorithm returns *true*. This can be seen as follows: if X is not Gorenstein, then $\mathcal{K}_X + m\mathcal{L}$ is not a Cartier divisor class; in particular, it is not base point free. Now assume that X is Gorenstein. Since the embedded monoid $\mathrm{BPF}(X) \subseteq \mathrm{Pic}(X)$ is spanning, we can apply Algorithm 4.7.7 and compute a point of its conductor ideal. Lemma 4.9.7 shows that we can bound m by $\nu - 1$; Lemmata 4.9.8 and 4.9.10 prove that the sets Gp_i^k, $1 \le i \le r$, $1 \le k \le \lfloor \frac{\alpha_i - 1}{m} \rfloor$, serve as validations sets of Cartier divisor classes. □

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